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Comments on the life and mathematical legacy of Wolfgang Doeblin

Bernard Bru[1], **Marc Yor**[2]

[1] Laboratoire de Mathématiques appliquées, Université Paris V, 45 rue des Saints-Pères, 75270 Paris Cedex 06, France
[2] Laboratoire de Probabilités et Modèles aléatoires, CNRS-UMR 7599, Universités Paris VI et VII, 4 Place Jussieu, Case 188, 75252 Paris Cedex 05, France (e-mail: deaproba@proba.jussieu.fr)

Abstract. This article contains the translation into English of the main results found in the Comptes Rendus Volume of December 2000, dedicated to Wolfgang Doeblin's sealed envelope sent to the Académie des Sciences de Paris in February 1940. The genesis of these results – both from human and scientific perspectives – is discussed, as well as their importance in our present understanding of one-dimensional diffusions.

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Introduction

In May 2000, the sealed envelope sent in February 1940 by Wolfgang Doeblin from the front line in Lorraine to the Academy of Sciences in Paris, was finally opened. This was a long-awaited event for researchers in probability, with some interest in the history of their field, and who had in the past been struck by the modernity of the ideas of Wolfgang Doeblin.

Once again, the Pli turned out to contain some gems, e.g. an extremely advanced representation of the standard one-dimensional diffusions. Apart from its purely scientific interest, the Pli reveals a lot about Wolfgang Doeblin as a human being fully involved in the second World War and torn, as is his whole family, between France and Germany.

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The Pli has now been published in its entirety in the *Comptes Rendus* of the Académie des Sciences [14] as a Special Issue, dated December 2000, and this seems to have awakened or renewed interest in both Wolfgang Doeblin's life and work. Perhaps as a consequence, Professor Sondermann has kindly asked us to present an English translation of selected pages of the Pli as well as some answers to the main recurring questions recently asked about Wolfgang Doeblin.

Here are the results of our efforts towards this goal. These are summarized in our Plan of the article:

1. About Plis cachetés in general, and the Pli no. 11.668 in particular.
2. The lives of Wolfgang Doeblin and Vincent Doblin; the phoney war; Vincent as mathematician – soldier – telephonist.
3. Main results found in the Pli; where does the Pli stand among studies of stochastic processes?
4. Selected pages from the Pli.
5. Reading Notes from the Pli.
   Bibliography.

The contents of these five sections were strongly influenced by the great number of questions asked, and by the reactions during our writing and immediately after the publication of the Comptes Rendus volume: What is a Pli cacheté? How did the Döblin family live before and after the war? How important are the contents of the Pli?

In fact, interest around the Pli seems to have gone way beyond the community of probabilists, due to a number of facts:

– world-wide media coverage was given to the announcement of the opening by the Académie des Sciences of a Pli cacheté deposited sixty years ago as well as to the tragic story of the life and death of Wolfgang Doeblin during WWII.
– people interested in the writings of Alfred Döblin also became curious about his son's life during WWII, if not his mathematics... One knows how much intertwined the lives of the young Wolfgang, an exceptionally gifted mathematician, and of his father Alfred Döblin, one of the great German writers of the 20th century, have been. Wolfgang was the beloved son of his mother Erna Döblin, who always kept his letters with her. Wolfgang's physical resemblance to his father was astonishing; they also had the same passion for poetry and music. Yet, their relationship was a source both of love and conflict. Some of Wolfgang's manuscripts which are deposited in the literary archives of Marbach are written on the back of Alfred's manuscripts. Paul Lévy, one of Wolfgang's mentors, was at the same time a friend of Alfred, and one of his daughters was a friend of Erna. Many literary critics have recognized in Edward Alisson, the hero of Alfred Döblin's last novel (Döblin 1966), not only a double of Wolfgang but also of Alfred. Edward who has been gravely wounded during the war, tries to escape the dark world which surrounds him, and is only liberated through the death of his parents... This

leads naturally to the grave where Wolfgang, Alfred and Erna are buried in the Vosgian village of Housseras where Wolfgang ended his life.

– some probabilists and/or physicists discovered that the genesis of a part of their field had escaped their attention. The Pli appears as an opportunity for looking back on their common past.
– but, perhaps, more important than anything else, Wolfgang Doeblin's figure stands out throughout the writing of the Pli and, in fact, throughout his whole life as an incarnation of "mankind's indomitable thirst for knowledge", to borrow a line from D. Williams.

Here, we have tried to respond simultaneously to these different interests. This task took us into a number of divergent directions, and we may only have fulfilled our task partially...

We warmly thank Professor Sondermann for his sustained interest in the development of our undertaking, and his judicious suggestions.

The two younger brothers of Wolfgang, Claude and Stephan Doblin, were very enthusiastic about our undertaking, and helped us to correct several erroneous facts. Nick Bingham, Ron Doney, Torgny Lindvall, Anne Ruston carefully read our manuscript, and suggested a number of improvements. We are very grateful to each of them.

1 About Plis cachetés in general, and the Pli No. 11.668 in particular

1.1 What is a Pli cacheté?

The procedure of a "Pli cacheté" goes back to the very origin of the Académie des Sciences. One of the first known examples was that of the deposit by Johann Bernoulli, on February 1st, 1701, of a "sealed parcel containing the problems of Isoperimetrics so that it be kept and be opened only when the solutions of the same problems by his brother, Mr. Bernoulli from Basle, will appear". A "Pli cacheté", since that time, allows an author to establish a priority in the discovery of a scientific result, when he/she is momentarily unable to publish it in its entirety, in a manner which prevents anybody from exerting any control, and/or asking for some paternity, over the result. This procedure continued after the creation in 1835 of the Comptes Rendus de l'Académie des Sciences which play a comparable role (to the Plis cachetés), but which, to some degree, are submitted to the judgments of peers and referees, while they do not allow in general the development of methods and proofs.

This procedure is still in use today and is subject to rules updated in 1990. These stipulate that a Pli can only be opened one hundred years after its deposit unless the author or his/her relatives explicitly demand it. Once the century has elapsed, a special commission of the Academy opens the Pli in the order of its registering and decides whether to publish it or not.

From here on, we shall refer to Doeblin's Pli cacheté No 11.668 as *the Pli*.

1.2 Why did Wolfgang Doeblin use this procedure?

One may ask about the reasons which led Wolfgang Doeblin to have recourse to the procedure of a Pli cacheté for his study of Kolmogorov's equation.

We go back to February 1940. Spring was approaching and with it a predictable German offensive. Wolfgang Doeblin, a soldier in the French Army, did not have time to finish writing up his results. He could not send a memoir in this state. He was lacking references, he needed to read again the whole manuscript and to complete the proofs. This would perhaps have meant one month's work given the rhythm Wolfgang Doeblin was able to furnish.

On the other hand, Wolfgang Doeblin had so far not published anything on the general case of Chapman's equation, which he had been studying since 1938. He decided to stop there the work begun in November 1939 and to concentrate instead on the writing of notes announcing his results in the general case. What should he do with his manuscript? He might send it to Fréchet or Lévy, but neither was entirely reliable. In 1938, Lévy kept in his filing cabinet the manuscript of the memoir [11] on the metric theory of continued fractions which Doeblin had asked him to present to Compositio Mathematica. Fréchet, on his side, was overwhelmed with diverse tasks and tended to forget things...Indeed, he was to forget in his papers, in turn, the last two notes written by Doeblin on Chapman's equation. These will only be published in the Blaubeuren volume as late as 1993 (Cohn 1993).

Moreover, Doeblin knew that the subject of Kolmogorov's equation was attracting a lot of interest and he feared to be preceded or plagiarized by someone...Of course, he could "chop" his manuscripts, sending the fully prepared part to a journal and the rough remaining part to his brother in the United States. This he had done before for his study of the set of powers of a probability ([13], 1940). In the end, however, the procedure of a Pli cacheté turned out to be the safest and most speedy way. Time was running out... Already, during the summer of 1938, when international tension was rising around the rape of Czechoslovakia, Doeblin had tried to safeguard his yet unpublished papers. In fact, before going hill walking in the Jura and the Alps, Wolfgang Doeblin had deposited two Plis cachetés (Nos 11.445 and 11.446) which he claimed back and recovered in their sealed forms on September 28th, the day before the signing of the Munich agreement.

Doeblin's case was not unique; other scientists were making use of the same procedure during the troubled period of the years 1938–1940. In particular, the work of Dedebant, Wehrlé and Kampé de Feriet on the statistical theory of turbulence was deposited in four Plis cachetés. Likewise, the theory of nuclear fission of Joliot, Halban and Kowarski was deposited in several stages between 1939 and 1940. Thus, some of the best kept atomic secrets during WWII may have been those kept in the attics of the Institute as well as a few ingenious proofs of the quadrature of the circle, the plans of several machines inducing never-ending motion, and Kolmogorov's equation.

Cases of Plis coming from the Army's postal sectors were more exceptional. Apart from the Pli 11.668, only two other Plis came from researchers drafted to the Army. One came from René Marconnet; this Pli has not been withdrawn, and we do not know anything about it. The other one was deposited by René de Possel, one of the founding members of Bourbaki, with H. Cartan and A. Weil. This Pli, the content of which is also unknown, was returned to its author on August 22nd 1947.

Thus, in February 1940, Doeblin finally decided to have recourse again to the procedure of a Pli cacheté. More than ever, he was anguished with the idea of dying whilst the results of his research about Kolmogorov's equation would forever remain unknown. Consequently, he resorted to two further precautions: in a letter dated March 12th 1940, he alerted Fréchet about sending the Pli and, in a separate mail registered on March 13th 1940, he sent to the Académie a copy of his memoir. He believed that the war was not going to last long and that he would be able to reclaim his manuscript, as he had done at the end of the summer of 1938, or that Fréchet would do it for him. All seemed to be well planned, except for what actually happened.

Doeblin died on June 21st 1940. His manuscripts lay scattered in several places – in Philadelphia, with his brother Peter, where a second full manuscript on the set of powers [13] as well as the rough draft of his general theory of chains [12] had been deposited, – in Paris, in the caves of the Sorbonne, with the papers of his father, other rough drafts and personal papers, – with Fréchet, two projects of Notes for the Comptes Rendus, – finally, in the Académie, the Pli cacheté 11.668.

But the war lasted for five years. Lévy was forced to go into hiding under a false name. He also needed to have recourse to the procedure of Pli cacheté (1943) for other reasons, which the Académie had not foreseen, namely racial banning. After the liberation of France, life was difficult. Fréchet, who had just lost his wife run over by an American military vehicle, was mainly preoccupied with the material needs of his grandchildren. He had clearly forgotten about the Pli and Doeblin's notes (in fact, Doeblin's death was only officially known as late as the Spring of 1944). Fréchet was no longer interested either in the theory of chained events or in Chapman's equation, the latter domain being now reserved to Lévy who was preparing an important book on the subject (1948). Nonetheless, Fréchet had not forgotten about Doeblin altogether. During the "Congrès de la Victoire de l'Association Française pour l'Avancement des Sciences" held in Paris at the end of October 1945, it was Fréchet, the holder of the Parisian Chair of Probability Theory and Mathematical Physics, who presented the bulk of the French work in Probability and Statistics undertaken during the German occupation. His lecture opened with a moving homage to the memory of Wolfgang Doeblin, "of German origin, but who became French before the War". Fréchet writes: "From all his soul, he (W. D.) wanted, when the war broke out, to show his gratefulness to his adopted fatherland, by fighting hard for her. This is the imprint which my conversations with him left me. But, it is in this ardent fight that he will meet his death on June 22nd 1940 (sic). One must hope that it will be possible to find the mathematical papers sent by Wolfgang Doeblin to his relatives in America and, in any case, to present a general study of the rich sequel of works which he published in a few years interval." (Fréchet 1947, p. 107).

Fréchet and Lévy involved themselves actively in the publication of Doeblin's last manuscripts. They cannot be accused of negligence, lack of interest or malevolence. It seems obvious that they would have edited the memoir on Kolmogorov's equation, had they known about it as there was every indication that it might become a "classic". In his presentation of the state of probability theory in France (1947), Fréchet alluded to the two C. R. Notes [CR9, 10] in a way that suggests that neither did he remember much from them nor had any recollection either of Doeblin's correspondence on this topic or of the Pli cacheté, nor of the CR Notes. As to Lévy, in his study of Doeblin's work (1955), he merely recalled the local theorem of the iterated logarithm for the regular movements contained in the Note [CR9] and concluded: "The premature death of the author prevented him from developing this note. Despite the few pages devoted to these questions by P. Lévy (1948, pp. 75–78), this note and those that had followed should doubtless inspire some further research"; (1955, p. 111).

In 1957, separated by a few months, Alfred and then Erna Döblin died. Their three remaining sons Peter, Claude and Stephan entrusted all their father's manuscripts to the literary archives of Marbach. Wolfgang's personal papers and various rough drafts, which had been kept by Erna Döblin, were recovered by Claude Doblin who would deposit them in Marbach at the end of the eighties.

Fifty years have now passed. A conference in honor of Wolfgang Doeblin was organized in the Institut Heinrich Fabri in Blaubeuren in Germany, November 2nd–7th 1991. This conference was initiated by K. L. Chung, planned by A. Blanc-Lapierre, H. Cohn, J. Gani, H. Hering and M. Iosifescu and chaired by J. L. Doob. This event naturally gave occasion to the renewed study of the unfinished work of Wolfgang Doeblin and to looking back upon the unpublished manuscripts (see Cohn 1993).

Fortunately, the archives of Maurice Fréchet had been deposited by his family in the Académie des Sciences. Fréchet had kept everything, his school exercise books, the manuscripts of his publications, all the reprints sent to him covering all kinds of topics, the text of his conferences, the notes of his university lectures in Strasbourg and Paris and, of course, all the letters he received during his very long academic life from dozens of scientists, from the Moscow school, the Polish school, from Romania, Bulgaria, Czechoslovakia, Yugoslavia, Greece, United States, from the main (non-German) analysts of the 20th Century, but also from statisticians, Fisher, Neyman, the Pearsons (father and son), and from many French scientists, amongst whom Lévy (whose extensive correspondence is being edited by B. Locker in his thesis 2001) and, of course, Doeblin. In Doeblin's correspondence with Fréchet, which amounts to about twenty letters, one finds Wolfgang's letter dated 12th of March 1940 announcing the dispatch of his Pli about Kolmogorov's equation. From there, it was easy to verify that the Pli had never been opened, and remained in the Archives of the Académie des Sciences.

Meanwhile, the important work of Alfred Döblin, which had so far been neglected, was gaining more interest and recognition. The publication of his complete works was undertaken; this involved, as often is the case, some difficult negotiations and complex rulings. This led to delays for the opening of the Pli. Claude Doblin was finally able to present this demand at the Académie in May 2000, and the Commission, in agreement with the 1990 ruling, undertook the opening of the Pli 11.668. The long night of the last manuscript of Wolfgang Doeblin had come to an end.

2 The lives of Wolfgang Doeblin and Vincent Doblin; mathematician–soldier–telephonist

2.1 Wolfgang Doeblin was born on the 17th of March, 1915, in Berlin[1]. His father Alfred Döblin (1878–1957), who belonged to a Jewish family, was a physician and was starting to get a name in the vanguard of German literature. He became famous in 1929 once his novel *Berlin Alexanderplatz* was published. The Döblin family was forced into exile in March 1933 after the burning of the Reichstag and the vote of full powers to Hitler. After a short stay in Zürich, the Döblins settled in Paris. Wolfgang, having finished his "humanities" in a Protestant Gymnasium in Berlin, enrolled for the Licence de Mathématiques at the Sorbonne in the autumn semester 1933. At the end of 1935, he carried out research about the theory of Markov chains under the guidance of Maurice Fréchet. At that time Paris was, with Moscow, one of the main mathematical centers interested in the new theory of probability. There one would meet Borel, Darmois, Fréchet, Lévy, Francis Perrin, and also a group of young mathematicians Dugué, Fortet, Loève, Malécot, Ville,..., each of whom was to defend his mathematical thesis bearing upon some probabilistic themes at the end of the thirties.

[1] The correspondence and the autobiography of Alfred Döblin (1970, 1980) give a clear picture of the peregrinations of the Döblin family in Europe and in the United States during the war. Particularly worth reading is Alfred Döblin's description of the French rout in June 1940 when two of his sons are fighting in French uniforms against the German troops, and another son is fighting in German uniform against the French troops.

While in Paris, Alfred Döblin continued his literary work. During the phoney war, he belonged to the Board of French propaganda. In July 1940, he managed to leave France together with his wife and their younger son born in 1926. The Döblins spent the war in the United States where Alfred Döblin found it hard to emerge. Alfred Döblin and his wife as well as their sons Peter and Stephan converted to catholicism at the end of the year 1940, which separated them even more from the Jewish community from which Alfred had parted soon after the end of WWI. The Döblins came back to France in 1945. Alfred Döblin started work for the cultural services of the French occupying forces in Germany, then stationed in Baden-Baden. His role consisted in reading the manuscripts presented by the German writers and journalists in order to obtain a publication visa from the French occupying forces. These activities of a censor dressed in the uniform of a French colonel did not help Alfred Döblin to re-enter the realm of German literature. His last years proved particularly difficult. He went back to live in Paris in 1952, and he will die in Emmendingen (Schwarzwald) in June 1957, almost completely forgotten, ignored by the different communities to which he had successively belonged – a Jew from Stettin who became a Parisian catholic, a Berlin physician who became a cosmopolitan writer, a banished European, but indeed a "sower and forerunner of the true European of the future, and a true citizen of the World", as his friend Hermann Kesten put it. See also (Lindvall 1991, 1993).

The young Doeblin very quickly obtained some most remarkable results. In a few months, Doeblin's name was to become well known to the small group of mathematicians interested in a theory which by then was in full bloom. In order to give an idea of the difficulty and the originality of the work achieved by Doeblin in so short a time and at such a young age, Paul Lévy (1955) compared Doeblin to Galois and Abel. Of course, one may argue about Lévy's judgment, but it is difficult to deny that Doeblin, together with Kolmogorov, Khinchin, and Lévy himself, was one of the main characters involved in probability theory in the thirties. At the age of 23 years and with only two years of active research behind him, Doeblin's performance must be considered unique, probably since Laplace[2].

Wolfgang Doeblin, together with his parents and his two younger brothers Claude and Stephan, acquired French citizenship in 1936. After defending his famous thesis in Mathematics [5] in Spring 1938, he was enlisted for two-years military service which had been deferred for the duration of his studies. At the beginning of November, he was posted to a battalion of the 91e RI stationed in Givet in the French Ardennes. Getting depressed by the barracks routine life, he stopped all his mathematical work for four months. It was only at the end of February 1939 that, in order to overcome his lethargy, Doeblin resumed his work. He had several themes in mind, among them Chapman's equation – which we shall discuss later – about which he had already written two Comptes Rendus (Acad. Sci. Paris) notes before his departure for Givet, as well as questions relating to independent random variables, which he had put aside in February 1938. He started working again on this second topic, and it was indeed in Givet, whilst listening with one ear to some courses for corporal students[3] that he wrote his fundamental memoir about the sets of powers of a probability law [13] which contains the theory of domains of partial attraction (i. e. the closure in law of the normalized powers of a given law), in particular, the "universal" laws which belong to the domain of partial attraction of all the infinitely divisible laws (see Feller 1966 for a presentation of this theory). We know from his correspondence (Cohn, p. 27) that, in July 1939, Doeblin was at last able to characterize the sets of infinitely divisible laws (the empty set, the Gaussian laws,..., all the infinitely

[2] About the scientific work in general of Wolfgang Doeblin, one may consult (Lindvall 1991; Lévy 1955; Cohn 1993; Chung 1964, 1992; Doob 1953; Feller 1950, 1966; etc.

T. Lindvall (1993, pp. 55–56) quotes K. L. Chung's review of (Lévy 1955) in the *Math. Reviews*: "After all there can be no greater testimony to a man's work than its influence on others. Fortunately, for Doeblin, this influence has been visible and is still continuing. On limit theorems his work has been complemented and completed by Gnedenko and other Russian authors. On Markov processes it has been carried on mostly in the United States by Doob, T. E. Harris and the reviewer. Here his mine of ideas and techniques is still being explored."

[3] Monsieur Paul Beaujot from Fromelennes in the Ardennes was posted to Doblin's company in Spring 1939. He remembers very well the soldier Doblin with whom he attended the course for corporal students in the battalion. Doblin made good use of the theoretical courses by writing the second part of his memoir on the sets of powers of a probability. He was often called to order as he was doing computations with no relation to the theory he was supposed to be learning. Nonetheless, he was the major of his group during the final examination in August 39. Vincent Doblin was always standing on his own, lost in his thoughts and computations, of which he could not speak and which have mostly been lost.

divisible laws) which constitute the domain of partial attraction of a given law. This necessary and sufficient criterion, about which Doeblin wrote that it is the most difficult problem he ever solved apart from the general theory of chains [12], has never been published or even clarified. Indeed, no one has been able to decipher the rough draft he sent to his brother Peter in the United States, and which was deposited after the war by his mother Erna Döblin in the Académie des Sciences where it has remained to this day... As usual, Doeblin made sure that his text would be illegible to anybody but himself, and to start understanding it, one would need, at least, a statement of the theorem he was proving, or some modern analogue which does not seem to exist. These details are only given to indicate the level of difficulties at which Wolfgang Doeblin was working, cut off as he was from every scientific contact. We also hope that these details may raise some new interest on "the last theorem of Givet".

Immediately after the outbreak of war, Vincent Doblin was incorporated into a new regiment, the 291e RI, attached to the "Secteur défensif des Ardennes", and stationed in the small village of Sécheval, south of Givet. His company's duty was to organize the defense on the Meuse between Anchamps and Château-Regnault, in the meanders of the Meuse, one of the most beautiful landscapes of the Ardennes. However, these beautiful autumn days seemed to accentuate the recurring low spirits of the soldier Doblin who for two months abandoned all idea of scientific work, only contenting himself with the correction of the galley proofs of his memoirs to be published, in particular [10] and [11], which his mother had sent him. Monotonous days were filled by Morse training with radios, shifts at the telephone booth of the battalion, drilling exercise with the section. The evenings were spent in a nearby farm where he went to drink some fresh milk, before going to sleep on the straw in an old kitchen which had been converted to a dormitory for fifteen soldiers[4]. In any case, the possibilities of intensive intellectual work were quite limited. Doeblin had no scientific document at hand and no place to work apart from the telephone booth. Even there, he could only find quietness during his night shifts. He was alone, hibernating. He no longer wrote to his parents. Maurice Fréchet, who had had no news from him since the beginning of the war, learned about his postal sector from his mother. He wrote to Doeblin asking him to collaborate on the scientific work which he directed at the Institut Henri Poincaré, then in the service of National Defense. This letter, which we have not yet retrieved, seemed to have had a beneficial effect on Doeblin's morale, since on October 29th, 1939 Doeblin gave a positive answer to Fréchet's invitation. Not long afterwards, in a letter dated November 12th, Doeblin informed Fréchet that he had started work again "oh! not much, about

[4] The building where he was staying belonged to the family Canot, who had been living in Sécheval for many years. Émile Canot, then 14 years old, very well remembers the soldier Doblin who used to come to the farm around 8 pm to drink some fresh milk. Émile Canot particularly remembers a discussion which struck him: one evening, Vincent Doblin told them in confidence that he was a Jew, and that he would never accept to be a prisoner of the Germans, and that he always kept on him a bullet to kill himself if he was captured. This verbal account is corroborated by others and serves to underline Doeblin's determination as well as his premonition of what was going to happen, a feeling that was deep inside him all the time.

one hour every day" and that he was writing the developed proofs of his Note on Kolmogorov's equation [CR9], which had been published a little before his departure for military service (Cohn pp. 29–28, 53–54). He was trying very hard, as he wrote to Fréchet, to "fight against depression. As I am not interested in alcohol, I cannot resort to getting drunk." Mathematics as a therapeutic against the blues, a nice Pascalian theme.

2.2 At this point, it may be useful to pause for one instant in order to examine the genesis of the work of Doeblin on Chapman's equation. It seems that it was only during the year 1937, after having completed his general theory of chains [CR5], [12], that Doeblin began in earnest his attack on one of the most important and most difficult problems of probability theory during the thirties (and following decades), the "Bernstein-Kolmogorov problem": given some local characteristics as general and natural as possible, to construct a movement whose law satisfies the functional equation of Bachelier-Smoluchowski-Chapman-Kolmogorov and to study its behavior[5]. By local characteristics, one must understand what determines the movement between the instants t and $t + dt$, that is, in the case of a continuous movement, the instantaneous speed of the non-random component (the drift), and the instantaneous variance of the random component (the

[5] Wolfgang Doeblin did not wait until 1937 to take on Kolmogorov's problem. He was one of those scientists who learn a theory while trying first to solve its open problems, preferably the most difficult ones, even if they do not understand its terms completely. We may try to establish an approximate date of Doeblin's first contact with Kolmogorov's problem. In the archives of Maurice Fréchet in the Laboratoire de Probabilités of the University Paris VI, one finds an abbreviated translation by Doeblin of the two memoirs of Kolmogorov (1931, 1933a). This is probably some work done on the request of his teachers, Fréchet or Darmois. Even a quick overall reading of these translations clearly shows that the young Doeblin was a beginner in probability theory, and more generally in Mathematical Analysis. For example, he translates the title of §4 "Das Ergodenprinzip" as "Le principe de l'Ergoden", obviously not knowing what this means, although since 1928, Hostinský, Hadamard, Fréchet had been using the term "principe ergodique" to denote the regular asymptotic behavior of Markov chains, by analogy with the ergodic property of dynamical systems. It may even be that Kolmogorov's "Ergodenprinzip" was a German translation of the "principe ergodique" in the theory of chains of Hostinský-Hadamard (1928), which Doeblin himself was to develop brilliantly from the beginning of the year 1936. It is then quite plausible that this is a text written at the beginning of 1936, or even more likely during the year 1935. It is known that during Spring 1934, Wolfgang Doeblin was a Licence student of G. Darmois who immediately noticed his extraordinary quickness of mind. From then on, Darmois regularly informed the young Doeblin about the most difficult open problems in the theory of probability. He may well have indicated to him Kolmogorov's problem, although this may have occurred with Fréchet, a close friend of Kolmogorov, who was in USSR during the fall of 1935. In any case, one finds in Doeblin's translation an important number of "typos" (more than in the original text in Math Annalen which Doeblin does not correct and which he does not appear to notice), or even mathematical errors of translation (for example, "totalstetig" on top of page 440 is translated as "totalement continue", whereas a more advanced student would have translated adequately as "absolument continue", showing there probably that he did not yet know Lebesgue's theory, in any case not better than that of Markov). Thus, it seems that, although not understanding it reasonably well, Doeblin would have read Kolmogorov's theory as early as 1935 when he was barely 20. He might then have decided that he was not quite ready and that he should rather begin to learn some analysis and to reconstruct in his own manner the theory of chains, which he will achieve from the beginning of 1936 in a very original manner. The astonishing maturity of the text of the Pli would thus seem a little better understandable. The Doeblinian theory of Kolmogorov's equation is certainly not the most difficult work realized by Doeblin in his very short scientific life, but in some way, it is the most mature and one of the most modern.

martingale part), or in the case of a discontinuous movement, the probability of going from one state to another during an infinitesimal interval of time (dt). This problem had been posed in one of the most famous memoirs of Kolmogorov published in 1931 in “the” Revue of *Math. Annalen*, and was soon followed by very important work of Khinchin, Petrowski and Feller. All these authors had used analytical methods borrowed from the theory of partial differential equations of parabolic type. The conditions they imposed on the local data were therefore analytic and seemed artificial as soon as the problem was considered from a probabilistic point of view: a random movement, which is continuous or discontinuous, without memory (non hereditary). Hence, as early as 1937, Doeblin’s idea was to solve Kolmogorov’s problem in such a way that the solution satisfied the following double criterion: “the local conditions which we impose must have a probabilistic meaning, a meaning for the movement, and the ideal solution will be a solution which allows us to read, in some way, the movement”.

In a program of research dated May 1937, Doeblin stated that he aimed to study some questions related to parabolic equations and he chose, in agreement with Fréchet, as a second thesis subject[6]: “Limit problems for the partial differential equations of parabolic type”, that is the theory which Khinchin, Petrowski and Feller had applied with success to Kolmogorov’s equation. All this indicates Wolfgang Doeblin’s strong and constant interest in this main research theme, whilst, at the same time (this is the year 1937), there was a steady stream of publications of his work on many other subjects : inhomogeneous chains, chains with complete links, general theory of chains, independent random variables, random continued fractions, ergodic theorem of Fortet-Doeblin-Yosida-Kakutani, etc.....He had also finished writing all of his thesis worked out the preceding year.

In October 1937, Doeblin took part in the Geneva Colloque on the Calculus of probabilities, and, upon this occasion, met all the leading (non-Russian) personalities of the theory, among them Feller, Hostinský, Cramér, and others. It was Doeblin who was asked by Fréchet to edit the conferences of Slutsky and Bernstein, who were absent from Geneva. The month of February 1938 was wholly devoted to the theory of independent random variables ([9], [CR6,7]). However, in March, he gave a main lecture on Chapman’s equation at the Hadamard Séminaire (see [14] for a transcription of this exposé); some of his future leading ideas are found here in seminal form. He defended his thesis on March 26th, 1938. The following three months were devoted to the preparation for the exam of General Physics which he still needed in order to obtain his “licence for mathematical teaching”[7]. This involved some quite intensive work which left

[6] For the French degree of “docteur ès sciences mathématiques”, there were two theses, the main one and a second one which was an oral examination whose aim was to test the breadth of knowledge and teaching abilities of the candidate (Taqqu 2001, p. 6).

[7] At that time, the licence for mathematical teaching consisted of three exams: Rational Mechanics, Differential and Integral Calculus, and General Physics. The latter exam was part of licences for teaching both mathematics and physics. Its program was encyclopedical covering all of classical physics. In June 1935, Doeblin brilliantly passed his examination in Differential and Integral Calculus. Since, during the previous year, he had passed the exam in Rational Mechanics and Probability Calculus (which is a specialized optional examination), he already held his “licence de doctorat”

very little time for personal research. It was during the summer of 1938, whilst walking alone in the Jura and the Alps and sleeping in Youth Hostels as he had done each summer since 1935, that Doeblin really worked on Kolmogorov's equation. It is difficult to date his results more precisely. The first results he obtained were probably those contained in the second note [CR10] and concerned, in the homogeneous case, the behavior of the movement in the neighborhood of a point where the local data vanish and its possible infinite branches as soon as the non random current is not compensated by the amplitude of the Gaussian movement. Most likely, this work had been motivated by the "stochastic" lecture of Bernstein in Geneva, which Doeblin had just edited (Bernstein 1938).

Whatever, in October 1938, shortly before his being drafted, Doeblin obtained his main results about Kolmogorov's equation. It is even possible that he presented part of them at the Séminaire Borel, although we have no definite proof of this. In a letter to Lévy written in that period and reproduced in (Cohn, pp. 38–39), Doeblin explained that he did not want to take on new topics of research, as he wrote: "I still have other things to write, and, above all, I am engaged in research about Chapman's equation which I would first like to finish, if only provisionally (it may keep me busy for my whole life[8])."

2.3 Thus, during the first fortnight of November 1939, in a small village of the Ardennes, as the glowing autumn receded giving way to a winter which promised to be severe, the soldier telephonist Vincent Doblin went out to buy a school exercise book of 100 pages and began to write down the development of his note "Sur l'équation de Kolmogorov" which he had written more than one year previously and never touched since, – one hour a day at most, and most likely during his night shifts in the telephone booth. The first pages of the Pli indicate that this was a form of therapy which the author imposed upon himself. The writing was relaxed, the hypotheses were not so precise, and the first proofs invoked "well-known arguments", but those were not made explicit. However, it appears that, as nights went by, the soldier Doblin was getting back into the game. The writing, as concise as ever, was becoming ethereal. The total absence of leaves had been forgotten, and around Christmas, the soldier Doblin was getting more and more addicted to his work. He even discovered some new results which amused him enough to incite him to write a second Note [CR12] on the same subject, before putting the last word to the writing of the memoir. It was now the beginning of January 1940 (Cohn, p. 29): it was extremely cold, the

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which enabled him to enrol for a thesis. This he did at the end of the year 1935. Doeblin was well aware that it would be difficult to obtain a University chair in France, as these were more or less "reserved" for the most brilliant French (native) students of the Ecole Normale Supérieure who all were holders of the complete licence for teaching and of "agrégation" (an exam with a plethoric program which gave access to the best teaching position in Lycées). In order to improve chances on his side, Doeblin estimated that he had to satisfy the French University rules and in particular to obtain the certificate of General Physics, which forced him to interrupt his research, at least while he learnt all of French physics in three months, of which he knew nothing. Indeed, Doeblin passed this examination in June 1938. In a letter to a Swiss friend dated July 1938, he confessed that this had been the hardest effort he ever had to make and that he was exhausted (Cohn, p. 41).

[8] In a recent paper on the Heat equation, Serge Lang used the same sentence!

ground was frozen to a depth of one meter, it was snowing, but the soldier Doblin was full of optimism. He may have dreamt, as the official French propaganda wished him to do, that the war was close to an end and that the 3rd Reich was imploding.

In the middle of January 1940, the dream was brutally replaced by reality, with the “alert on Belgium”. On January 11th, 1940, a Luftwaffe plane crashed on Belgian territory. Belgium was at this point a neutral country. The pilot was arrested and in his suitcase papers originating from the German Headquarters were found. These demonstrated that the Wehrmacht, far from collapsing, was getting ready to re-edit the Schlieffen plan of 1914 by attacking Belgium as soon as the weather would permit. In fact this plan was soon replaced by the Manstein plan with the main attack in the Ardennes, the attack on Holland and Belgium being only a trap to attract the Allied forces into Belgium, so that the net, once lifted, would eliminate the best enemy troops as far as possible. However, in January 1940, the papers found on the German pilot had the impact of a bomb. The French troops on the Belgian and Luxembourg fronts were put on alert and there were talks about entering Belgium. One witnesses a huge ballet of fighting units the logic of which was difficult to penetrate, but which, relevant to our case, resulted in the transfer of the 291e RI from the Meuse front to that of Lorraine. Hence, Kolmogorov’s equation moved from the Ardennes to Meurthe-et-Moselle under conditions of minimum comfort. From the report of Captain Camus, who commanded Doblin’s company, it appears that, on January 25th, the troop took a train to reach Rosières aux Salines, east of Nancy. The cold was intense. In order to allow the soldiers to warm up their shoes during the journey, fires were lit each time the train halted. Doblin’s company were marched to Athienville, a small Lorraine village close to Arracourt. The first weeks were difficult, with a complete battalion of more than 700 men billeted in a village of less than 150 inhabitants. Doblin slept in an attic with no heat, where it snowed, but, soon, his section was settled in relatively comfortable barracks specially built for the troop. Vincent Doblin was to remain there until March 14th, 1940. The 291e RI practiced before moving to the front line, and the quiet garrison life had started again.

Thus, it may well have been in Athienville, probably around the middle of February, that Doeblin finished writing the Pli. He would then have sent it to the Académie. This may explain the changes in numbering and pagination. There is an Ardennes pagination and a Lorraine pagination. Doeblin wrote to Fréchet that at some point he had had enough of Kolmogorov’s equation. He still had to write up the Note of 1939 [CR10] and the diverse results he had obtained since then. To establish a date, he sent, probably at the same time as the Pli, a second Note [CR12] which was presented by Borel on March 4th. It is difficult to give a more precise chronology in the absence of truly decisive elements. Rather than to carry on improving his manuscripts, he preferred to work directly on the mixed case of Chapman’s equation. The “local stochastic conditions” now allowed the movement to go at once from one state to another without solution of continuity (the probability of a sudden displacement of the movement X towards

L, between t and $(t + dt)$, is equal to $c\,(X\,(t)\,, L, t)\,dt$ for a local data c, satisfying some natural conditions), if not they are "regular" in the sense of the Pli. The question was then, for some given "local conditions" and under some adequate hypotheses, to determine the law of the movement which satisfies Chapman's equation in an "ideal form" which allowed one to follow its evolutions in the course of time. This work, which must have begun in February, was to continue up to mid-April. Hence, these two months were devoted to the general problem of Bernstein-Kolmogorov about which he had been thinking for about three years. It is likely that Doeblin wanted to finish an organized general scheme of work about Chapman's equation before the spring and a possible German attack. His spirits remained high, one reason being that, at long last, he may possibly have obtained a leave in the middle of March, which he may then have put to profit by going to the Institut Henri Poincaré (IHP) to look for the memoirs of Hostinský he needed.

During Doblin's leave, assuming it took place, the 291e RI left Meurthe-et-Moselle and was marched to the Defensive sector of the Sarre, on the Maginot line. The 3rd Battalion was stationed in Oermingen, Bas-Rhin, in barracks which had been built between 1936 and 1938 to house fortress troops guarding the sector's blockhouses. The 3rd Battalion was to remain in Oermingen till April 17th, 1940. During the stay of the 291e RI on the Maginot line, the soldiers of classes 30 and below were exchanged with younger soldiers from fortress troops. The soldier Doblin who belonged to the class 1935 remained in his battalion, but half of his comrades left, and a large number of officers was replaced. In particular, Doblin's company was entrusted to Captain François Renard, a priest from the diocese of Soissons. He became Doblin's immediate superior and was to remain so up to June 20th. Doblin, who seems to have been relatively aside in his company where his work on Kolmogorov's equation may not have found too much interest, was getting even more isolated from his fellow soldiers whom he did not know. Yet, this did not seem to affect his enthusiasm for work, and it was in Oermingen that he wrote three drafts of Notes on Chapman's equation. Only one would be published, through the help of Fréchet, [CR13], which was presented by Borel on April 29th, 1940. As we mentioned above, Doeblin's goal was to determine the "ideal form" of the solution of Chapman's equation which correspond to some given local conditions with possible jumps; it is clearly given by a series of multiple integrals with increasing orders as in Hostinský and Feller, but each of the terms of the series now had a clear probabilistic meaning.

The two other drafts of Notes written in April 1940 remained in the papers of Fréchet and were only published in the Blaubeuren volume (Cohn, pp. 32–36). At the end of his last letter to Fréchet, he announced other results to come on the control of small jumps, but these results have not been found, and most likely, were never written (Cohn, p. 36). Likewise, the proofs of the results in the mixed case are not available. The last theorems of the Maginot line are still resisting.

On April 17th, 1940, Doblin's regiment moved to the front line, close to the German frontier on the loop of the Blies, between Sarreguemines and Bliesbruck. The 3rd Battalion's HQ was established in Folpersviller, east of the airfield at

Sarreguemines. Wolfgang Doeblin found again the – completely evacuated – town where he had spent his early childhood from 1915 to 1917, when his father had volunteered as a physician in the Military Hospital of Saargemünd, the German Sarreguemines. In fact, his brother Klaus was born in Saargemünd. Doeblin seemed confident as to the end of the conflict, and was interested in getting a grant for his return to civilian life. This he estimated to be likely before the end of the year 1941. However, despite his optimism, Doeblin no longer found time nor possibilities for work. He sent back the reprints which Fréchet had communicated to him, with just a short postcard of thanks. It was dated April 21st, 1940. He was never to resume his work. For Wolfgang Doeblin, the study of Chapman's equation had ended.

The German offensive began on May 10th, 1940. The Sarre frontline where the 291e RI was positioned came under intensive bombardment. Under enemy fire, the soldier Doblin restored interrupted contacts, an act for which he was decorated with the Croix de Guerre and on which occasion he proved his great physical courage and disdain of death. This he was demonstrate even more between the 14th and the 20th of June when his regiment was trying to slow down the progress of the First German Army in its move south.

During the night of June 20th to 21st, as the remains of his decimated regiment are in the Vosges, completely encircled by German troops and surrender is imminent, the soldier Doblin who, according to the opinion of his superiors, has always been a "constant model of bravery and devotion", leaves his company and tries to escape on his own. After walking all night long, he finds himself inside the German net in the village of Housseras which has been taken by advanced elements of the 75th German Infantry Division. Refusing to give himself up as a prisoner, he then shoots himself in the head[9]. The life of the soldier Vincent Doblin stops here. He is buried in the evening of the 21st, without ceremony, without name, without a coffin. His body was only to be identified in April 1944[10]. A plaque has just been erected there, on June 2nd, 2001, in commemoration of Wolfgang Doeblin and his study of Kolmogorov's equation.

[9] The suicide of Wolfgang Doeblin has been described by several trustworthy witnesses. When he understands that the village of Housseras is surrounded by German soldiers, Wolfgang enters a farm, which belongs to the Triboulot family. There, without saying a word, he burns all his papers in the kitchen stove. He then comes out of the farm building, enters the barn and shoots himself in the head. Thus, Wolfgang Doeblin wanted to disappear in silence. Among his burnt papers, there may have been his "research notebook" in which he had always jotted down new questions to study, ideas to develop... and which has not been found. The Nazis had burnt the works of his father and had forced his family into exile. For Wolfgang Doeblin, there remained the ultimate freedom to burn his papers himself and to kill himself in order to preserve his ideal of life and the beauty of his work.

[10] The identification of Vincent Doblin's body was made possible thanks to the research undertaken by Marie-Antoinette Tonnelat as early as July 1940. Marie-Antoinette Tonnelat completed her university studies in IHP at the same time as Wolfgang Doeblin; she soon became his best (and only true) friend. Specialized in theoretical physics, she was to defend in 1941 her thesis written under the direction of Louis de Broglie. One may refer to (Cohn, pp. 45–46) for more details about this topic.

3 Where does the Pli stand among studies of stochastic processes?

3.1 One may invoke many reasons why the emergence of a specific branch of probability – the study of stochastic processes – took a quite tortuous path throughout the XXth century.

On one hand, the pioneers were very often quite original mathematicians, such as Bachelier, Lévy, Itô,..., whose novel ways of looking at things took a long time to be accepted.

On the other hand, perhaps the fact that Brownian motion possesses so many properties, which we summarize as:

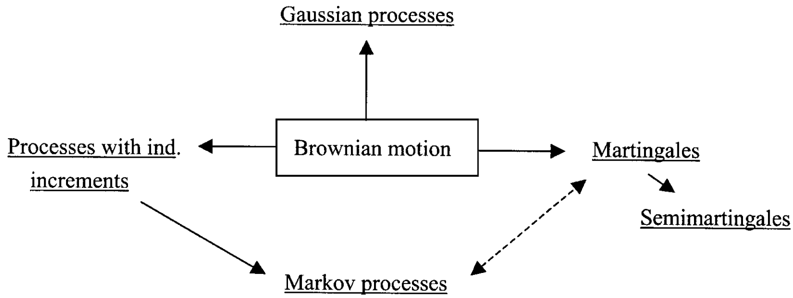

Fig. 1. Brownian motion and related processes

led many authors to develop studies of one or another special class of processes, thus giving a hard time to outsiders...

The Pli is mostly concerned with the construction and study of continuous Markov processes, particularly of one-dimensional diffusions.

3.2 The studies of these processes, prior to WWII, were done exclusively with the help of differential equations – more precisely, second order parabolic PDE – and pathwise constructions of these processes do not appear as such, although they are clearly present in the thoughts of Kolmogorov, Bernstein, Feller...

All along, from 1940 onwards, there has been a constant evolution from *PDE arguments* which we may call "exterior calculus", in that it expresses as a solution of a PDE the expectation, say:

$$u\,(t,x) = E_x\,[F_t]$$

of the (Brownian) functional F_t under P_x, the law of Brownian motion or of a diffusion starting from x, to *pathwise arguments*, which we might call "interior calculus", indeed, essentially Itô calculus, consists of the study of the process $(F_t, t \geq 0)$ "inside" the expectation E_x, e.g.: its semimartingale decomposition.

Again, this has not been a perfectly linear trend, as the following example illustrates: the paradigm of exterior calculus may well be the *Feynman-Kac formula* (1949), which characterizes the function:

$$\nu\,(x) = E_x\left[\int_0^\infty dt\,\exp\left(-\lambda t - \mu\int_0^t ds\,f(B_s)\right)\right]$$

as the unique solution of a certain Sturm-Liouville equation.

A very nice application, given by Kac, has been the recovery of the arc sine law of Paul Lévy for Brownian motion:

$$P_0\left(A_t \equiv \frac{1}{t}\int_0^t du\, 1_{(B_u>0)} \in da\right) = \frac{da}{\pi\sqrt{a(1-a)}} \quad ,$$

which Lévy obtained through a beautiful analysis of the Brownian paths (1939, pp. 317–320), as a consequence of the identity:

$$A_t \overset{\text{(law)}}{=} A_{\tau_s} \quad \text{(for fixed } t, s > 0\text{)},$$

where (τ_s) is the inverse local time.

Nonetheless, Itô's calculus took 25 years (1944–1969) to be accepted, the latter year being that of the publication of McKean's marvellous little book: *Stochastic integrals*. Itô's exposition of excursion theory (1970) represents a second generation of "interior calculus", where (Brownian) paths are decomposed into excursions away from a point, formalizing ideas which pervade Lévy's deep study of linear Brownian motion (1939, 1948). Again it takes about 10 years (1970–1980) to make this new tool operational, thanks to D. Williams integral representation of Itô's (σ-finite) measure of Brownian excursions, see e.g., L.C.G. Rogers (1981, 1989). This is followed by yet another wave (around 1983) where, through the works of Neveu, Pitman, Le Gall, Aldous..., trees and Branching processes are constructed from Brownian excursions.

Exterior Calculus	**Interior Calculus**
Feynman-Kac formula	Itô's calculus
Poisson's equation	Itô's Excursion theory
Resolvents	Trees and branching processes in Brownian excursions
Semigroups	
Infinitesimal Generators	

Fig. 2. Two complementary calculus

We should probably come back to matters which lie closer to the Pli: in Fig. 1, we have linked with a broken arrow: [Markov processes] and [Martingales]; this intends to mean that both studies are intimately linked: indeed, Stroock-Varadhan's introduction of a general martingale problem to characterize a given diffusion (1969, 1979) has been a very powerful tool, extending the martingale characterization of Brownian motion. As we shall see below, in Sect. 3.3, Wolfgang Doeblin was not far from this point of view.

To further insist upon the systematic evolution from Markov processes to martingales[11], we discuss succinctly the several developments of the so-called

[11] As Le Jan puts it in his paper "Martingales et changement de temps", *Sém. Proba. XIII*, Lect. Notes in Maths 721, pp. 385–399, Springer (1979), about time changes of martingales: "Going through the usual procedure, which passes via Markov processes, one may see that the "probabilistic version" of this formula is an expression of the predictable increasing process associated to the discontinuous part of a time-changed predictable martingale."

Girsanov's theorem, which explains how stochastic processes are being transformed under absolutely continuous changes of probability laws: starting with (Fortet 1943, Chap. II), Cameron and Martin (1945) who related the laws of Brownian motion and Brownian motion with drift, such results are extended to diffusions by Maruyama (1955) and then Girsanov (1960), and then found a simple and very general, almost "ultimate" extension with the Van Schuppen-Wong formulation (1974); if (M_t) is a (F_t) martingale under P, and $Q \ll P$, then (M_t) is a semimartingale under Q and its semimartingale decomposition(s) may be written in terms of bracket(s) involving (M_t) and $D_t \equiv \frac{dQ}{dP}\Big|_{F_t}$.

All previous formulations (in Markovian settings...) could then be easily recovered and understood from the Van Schuppen-Wong formulae.

$$\begin{tikzcd}
& \text{Fortet (1943)} \arrow[d] & \\
& \text{Cameron-Martin (1945)} \arrow[dl] \arrow[dr] & \\
\text{Maruyama (1955)} \arrow[dr] & & \text{Girsanov (1960)} \arrow[dl] \\
& \text{Van Schuppen-Wong (1974)} \arrow[d] & \\
& \text{Enlargement formulae (1977-2000)} &
\end{tikzcd}$$

(Barlow-Jeulin-Jacod-Föllmer-Imkeller-Amendinger ...)

Fig. 3. Evolution of Girsanov's theorem

3.3 As we shall see, Wolfgang Doeblin takes the martingale point of view in his analysis of the paths of an inhomogeneous real-valued diffusion $(X_t, t \geq 0)$, starting from x, with drift coefficient $(a\,(y,s))$ and diffusion coefficient $(\sigma\,(y,s))$, as follows:
(at this point, it is essential to recall that the general notion of martingale, and many facts about the structure of continuous martingales did not exist, when the Pli was written!)
i) $Z_t \stackrel{\text{def}}{=} X_t - x - \int_0^t a\,(X_s, s)\,ds$, $t \geq 0$, and $Z_t^2 - H_t$, $t \geq 0$, where $H_t = \int_0^t \sigma^2\,(X_s, s)\,ds$, are martingales; again: the term martingale is not found in the Pli, but the results are proven!
ii) there exists a Brownian motion $(\beta\,(u), u \geq 0)$ such that:

$$Z_t = \beta\,(H_t) \quad ;$$

in fact, Doeblin introduces the time change:

$$\theta\,(\tau) = \inf\{t : H_t > \tau\}\,, \quad \tau \geq 0,$$

and shows that: $\beta\,(\tau) = Z_{\theta(\tau)}$, $\tau \geq 0$, is a Brownian motion.

Finally, collecting i) and ii), Doeblin has obtained the representation of $(X_t, t \geq 0)$ as:

$$X_t = x + \beta_{H_t} + \int_0^t a\,(X_s, s)\,ds\ . \tag{1}$$

3.4 A few years later, K. Itô presented $(X_t, t \geq 0)$ in the form of the solution of a stochastic differential equation:

$$X_t = x + \int_0^t \sigma(X_s, s)\, dB_s + \int_0^t a(X_s, s)\, ds \ , \tag{2}$$

where $(B_s, s \geq 0)$ is a Brownian motion.

Then, the comparison of Doeblin's and Itô's representations (1) and (2) yields:

$$\int_0^t \sigma(X_s, s)\, dB_s = \beta(H_t)\ , \quad t \geq 0\ , \tag{3}$$

a result which would only be understood in a general setting years later with the Dubins-Schwarz and Dambis, both in 1965, representation of a continuous martingale $(M_t, t \geq 0)$ as:

$$M_t = \gamma\left(\langle M\rangle_t\right), \quad t \geq 0,$$

where $(\gamma(u), u \geq 0)$ is a Brownian motion.

3.5 Together with the representation (2), K. Itô establishes the key point of stochastic calculus: *Itô's formula* (1951b), which in this context, may be stated as follows:

if φ: $R_+ \times R \to R$ is $C^{1,2}$, then:

$$\varphi(X_t, t) = \varphi(x, 0) + \int_0^t \bar{\sigma}(X_s, s)\, dB_s + \int_0^t \bar{a}(X_s, s) ds \ , \tag{4}$$

where

$$\bar{a}(x, s) = \varphi'_s(x, s) + \varphi'_x(x, s)\, a(x, s) + \frac{1}{2}\varphi''_{x^2}(x, s)\, \sigma^2(x, s),$$

$$\bar{\sigma}(x, s) = \sigma(x, s)\, \varphi'_x(x, s)\,.$$

3.6 In the Pli, although Doeblin does not know, of course, about Itô's stochastic integral – a creation of Itô in 1942 – he established for his diffusions, once represented in the form (1), the following change of variables formula:

$$\varphi(X_t, t) = \varphi(x, 0) + \delta\left(\bar{H}_t\right) + \int_0^t \bar{a}(X_s, s) ds$$

where $(\delta(u), u \geq 0)$ is a Brownian motion, and:

$$\bar{H}_t = \int_0^t \bar{\sigma}^2(X_s, s)\, ds\,.$$

3.7 We hope that the above paragraphs – in this Sect. 3 – have given the reader the feeling that the Pli stands out as a link between the analytical researches of pre WWII, and the post WWII pathwise constructions.

It is now time for the reader to enjoy a few selected pages from the Pli, which we have translated into English.

Nonetheless, as T. Lindvall wrote to us, it must be "pointed out that this is not a matter of a finished manuscript. Reading one of the *published* papers by Wolfgang Doeblin is a pleasure: clear, carefully prepared, filled with a finely tuned enthusiasm. I regret if some reader's experience of his writing is limited to the Comptes Rendus pages or the English version of them; they get the wrong impression."

Some further discussion about the modernity of Doeblin throughout the Pli is made in Sect. 5.

4 Selected pages from the Pli

Research on Kolmogorov's equation

Definition of Kolmogorov's equation. Let us consider a particle which moves randomly on the line (or on a segment of the line). Assume that there exists a well defined probability $F(x,y,s,t)$ such that the particle which has been at time s in the position x is at time $t(> s)$ to the left of y, a probability which does not depend on the preceding movement of the particle. We shall assume that $F(x,y,s,t)$ is (B) measurable with respect to x, s and t, and that it solves the functional equation:

$$F(x,y,s,t) = \int_{-\infty}^{\infty} F(z,y,u,t)dF_z(x,z,s,u)$$

We assume furthermore that the following limits exist (except for "singular" points which vary continuously with time and are in a finite number in every finite interval):

$$\lim_{t\to s} \frac{1}{t-s} \int_{|y-x|<1} (y-x)dF(x,y,s,t) = a(x,s)\ , \tag{1}$$

$$\lim_{t\to s} \frac{1}{t-s} \int_{|y-x|<1} (y-x)^2 dF(x,y,s,t) = \sigma^2(x,s)\ , \tag{2}$$

$$\lim_{t\to s} \int_{|y-x|>\eta} dF(x,y,s,t) = o(t-s), \quad \text{for every} \quad \eta\ . \tag{3}$$

If x is a singular point at time s, one has only (3). We assume that the limits (1) and (2) take place uniformly and that $a(x,s)$ and $\sigma^2(x,s)$ are continuous with respect to x and s on every interval which does not contain singular points, (3) taking place uniformly with respect to x.

We assume furthermore that one has:

$$\lim_{t\to s} \overline{\lim_{x\to-\infty}} \left[1 - F(x,y,s,t)\right](t-s)^{-1} = 0 \tag{4}$$

$$\lim_{t\to s} \overline{\lim_{x\to\infty}} F(x,y,s,t)(t-s)^{-1} = 0 \tag{5}$$

If all these conditions are satisfied, we say that the movement is regular. We shall only consider regular movements.

. . .

Local Gaussian movement. Assume that $X(s) = x$, x being a regular point at time s, and $\sigma(x,s)$ being $\neq 0$.

Consider $[X(t) - X(s)]/\sqrt{t-s}$. We may write:

$$X(s+\Delta) - X(s) = \sum_1^n \left(X\left(s + \frac{i}{n}\Delta\right) - X\left(s + \frac{i-1}{n}\Delta\right)\right).$$

If Δ is sufficiently small, the probability for one of the quantities $\left|X\left(s+\frac{i}{n}\Delta\right) - X(s)\right|$ to be $> \varepsilon$ is $< \varepsilon/2$, and we can find n such that the probability for one of the $\left|X\left(s+\frac{i}{n}\Delta\right) - X\left(s+\frac{i-1}{n}\Delta\right)\right|$ to be $> \varepsilon\Delta$ is $< \varepsilon$. Denote:

$$\Delta_i = X\left(s + \frac{i}{n}\Delta\right) - X\left(s + \frac{i-1}{n}\Delta\right)$$

$$\begin{aligned} \text{If} \quad \left|X\left(s + \frac{i-1}{n}\Delta\right) - X(s)\right| < \Delta, \text{ let } \overline{\Delta_i} &= \Delta_i, \text{if} \quad |\Delta_i| < \varepsilon\Delta, \\ &= 0, \text{if} \quad |\Delta_i| > \varepsilon\Delta\ . \end{aligned}$$

If $\left|X\left(s+\frac{i-1}{n}\Delta\right) - X(s)\right| > \Delta$, we take for $\overline{\Delta_i}$ an arbitrary variable whose mathematical expectation is $a(x,s)\frac{\Delta}{n}$, and whose standard deviation is $\sigma(x,s)\sqrt{\frac{\Delta}{n}}$, with $E|\overline{\Delta}_i^3| < \varepsilon\frac{\Delta^2}{n}\sigma^2$.

We have for every $\pm X\left(s+\frac{i-1}{n}\Delta\right)$

$$\begin{aligned} E'\left[\overline{\Delta_i}\right] &= [a(x,s)+\eta]\,\frac{\Delta}{n} \\ E'\left[\Delta_i^2\right] &= \left[\sigma^2(x,s)+\eta'\right]\frac{\Delta}{n}\ \ldots \\ E'\left[\Delta_i^3\right] &< \varepsilon\Delta E\left[\Delta_i^2\right] \end{aligned}$$

where E'denotes the mathematical expectation evaluated if $X\left(s+\frac{i-1}{n}\Delta\right)$is determined, η and η' being extremely small. It then follows easily from (e.g. S. Bernstein *Math. Ann.* 1927) that the characteristic function of $\left[\sum_1^n \overline{\Delta}_i - \Delta a(x,s)\right]\Delta^{-1/2}$ converges, as $\Delta \to 0$, towards $\exp\left\{-\sigma^2(x,s)\frac{t^2}{2}\right\}$. $\sum_1^n \Delta_i$ differs from $\sum_1^n \overline{\Delta}_i$ only on sets of arbitrarily small probability; hence, if $\Delta \to 0$, the probability distribution of $\left[\sum_1^n \Delta_i - \Delta a(x,s)\right]\Delta^{-1/2}$ converges towards the centered Gauss distribution with standard deviation $\sigma(x,s)$.

We may say:

If $\sigma(X(s),s)$ differs from 0, the local displacement ΔX is the result of the superposition of a non-random displacement with speed $a(x,s)\,\Delta$ and a Gaussian random movement with zero mean, and whose standard deviation is $\sigma(x,s)\sqrt{\Delta}$ (we shall call $\sigma(x,s)$ the amplitude of the Gaussian movement).

Locally, the non-random movement, being of the order of Δ, is negligible with respect to the Gaussian movement. It is not so when Δ is no longer infinitely small. It is to be remarked that the decomposition of the movement, as indicated above, is not invariant when one makes a change of variables, even if the latter has the simple form $x' = \varphi(x)$, $y' = \varphi(y)$, $t' = t$, $s' = s$.

If $\sigma = 0$, $[X(s+\Delta) - X(s)]$ is, outside of cases of probability $\Xi(\Delta)$,[12] $0(\Delta)$.

...

6) Consider the process $Z(t) = X(t) - \int_0^t a(X(u), u)\, du$....

...[13]

VII. It follows from the preceding proof that under the hypothesis H':

$$E_{H'}(Z(t) - Z(s)) = 0 .$$

...

VIII. Let $\theta(\tau)$ the random time at which

$$\int_0^\theta \sigma^2 [X(u), u]\, du = \tau$$

resp. if $|X(t)| = 1$ for the first time, at a random time u_1, and that

$$\int_0^{u_1} \sigma^2 [X(u), u]\, du < \tau$$

the random instant at which:

$$\int_0^{u_1} \sigma^2 [X(u), u]\, du + \overline{\sigma}^2 [\theta(\tau) - u_1] = \tau$$

Lemma. *For any value of $Z[\theta(\Delta)]$, for $\Delta \le \tau$,*

$$E\left[Z\left(\theta\left(\tau'\right)\right) - Z(\theta(\tau))\right] = 0 \quad ,$$

$$E\left[Z\left(\theta\left(\tau'\right)\right) - Z(\theta(\tau))\right]^2 = \tau' - \tau.$$

Proof. We may write

$$Z(\theta(\tau)) = \lim_{n\to\infty} \sum_{i=1}^n \left\{Z\left[\frac{i}{n}\right] - Z\left[\frac{i-1}{n}\right]\right\}\varphi_i \quad ,$$

where $\varphi_i = 1$, if $\theta(\tau) < \frac{i-1}{n}$, $= 0$ if $\theta(\tau) \ge \frac{i-1}{n}$

$$E[Z(\theta(\tau))] = \lim_{n\to\infty} \sum_{i=1}^n E\left\{\left[Z\left[\frac{i}{n}\right] - Z\left[\frac{i-1}{n}\right]\right]\varphi_i\right\} \quad ,$$

and since

$$E\left[Z\left[\frac{i}{n}\right] - Z\left[\frac{i-1}{n}\right]\right] = 0$$

[12] Doeblin's notation $\Xi(\Delta)$ stands for any quantity converging to 0 as $\Delta \to 0$.

[13] To make paragraph VIII – which is essential in the Pli – understandable, we insert here the definition of Z in paragraph 6), and the statement of its martingale property, proven by Doeblin in paragraph VII. See our above Subsect. (3.3) for a succinct discussion of Z.

for all $Z(t)$ with $t \leq (i-1)/n$, one finds:

$$E\left[Z\left(\theta\left(\tau\right)\right)\right] \equiv 0\ .$$

and then $E\left[Z\left(\theta\left(\tau'\right)\right) - Z\left(\theta\left(\tau\right)\right)\right] = 0$, and we see easily that this formula remains true if one knows the values of $Z\left(\theta\left(\tau_i\right)\right)$ or of $X\left(\theta\left(\tau_i\right)\right)$, for any set of instants $\tau_i < \tau'$.

One also proves easily that

$$E\left[Z\left(\theta\left(\tau\right)\right) - Z\left(\theta\left(\tau'\right)\right)\right]^2 = \tau - \tau'.$$

IX. **Lemma.** *The probability law of*

$$\left[Z\left(\theta\left(\tau\right)\right) - Z\left(\theta\left(\tau'\right)\right)\right] \Big/ \sqrt{\tau - \tau'}$$

is the centered, reduced normal Gaussian for any j values of $Z\left[\theta\left(\tau'\right)\right]$.

Proof. The function $Z\left[\theta\left(\tau\right)\right]$ is a continuous function of τ, since $Z\left(t\right)$, $\theta\left(\tau\right)$ and $\tau\left(\theta\right)$ also are.

Indeed, one has:

$$A < \frac{\theta\left(\tau\right) - \theta\left(\tau'\right)}{\tau - \tau'} < B$$

if $\frac{1}{B} < \overline{\sigma}^2$, $\sigma^2(x,s) < \frac{1}{A}$. Hence, since $Z(t)$ is a. s. continuous, so is $Z\left[\theta\left(\tau\right)\right]$.

We may write

$$Z\left(\theta\left(\tau\right)\right) - Z\left(\theta\left(\tau'\right)\right) = \sum_{j=1}^{n} \left\{Z\left(\theta\left(\tau_j\right)\right) - Z\left(\theta\left(\tau_{j-1}\right)\right)\right\}$$

where $\tau_n = \tau$, $\tau_0 = \tau'$, $\tau_j - \tau_{j-1} = n^{-1}\left(\tau - \tau'\right)$.

Apart from cases of probability $< \overline{\varepsilon} = \varXi\left(n\right)$, $\left|Z\left(\theta\left(\tau_j\right)\right) - Z\left(\theta\left(\tau_{j-1}\right)\right)\right|$ will be $< \varepsilon$ for every j.

If we define

$$\begin{aligned} \varDelta\overline{Z_j} &= Z\left(\theta\left(\tau_j\right)\right) - Z\left(\theta\left(\tau_{j-1}\right)\right) \ \text{ if } \ \left|Z\left(\theta\left(\tau_j\right)\right) - Z\left(\theta\left(\tau_{j-1}\right)\right)\right| \leq \varepsilon\,, \\ \varDelta\overline{Z_j} &= 0 \ \text{ if } \ \left|Z\left(\theta\left(\tau_j\right)\right) - Z\left(\theta\left(\tau_{j-1}\right)\right)\right| > \varepsilon\,, \end{aligned}$$

$$\overline{Z} = \sum_{j=1}^{n} \varDelta\overline{Z_j} \quad ,$$

the functions $\overline{Z}$ and Z coincide outside of cases of probability $< \overline{\varepsilon}$. H being any hypothesis concerning $Z\left[\theta\left(\tau_i\right)\right]$, $i \leq j-1$, it follows from the previous discussion that

$$E_H\left[\varDelta\overline{Z_j}\right] = E_H\left[Z\left(\theta\left(\tau_j\right)\right) - Z\left(\theta\left(\tau_{j-1}\right)\right)\right] + o\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right)$$

and

$$E_H\left[\varDelta\overline{Z_j}^2\right] = \frac{\tau - \tau'}{n} + o\left(\frac{1}{n}\right),$$

$$E_H\left[\Delta\overline{Z}_j^3\right] < \varepsilon E\left[\Delta Z_j^2\right].$$

We may then apply a Proposition of M. S. Bernstein and we conclude that the law of $\sum \Delta Z_j$ converges to the law of Gauss if $n \to \infty$, and since Z and $\overline{Z}$ are identical outside of cases of probability $\Xi(n)$, the lemma follows.

X. **Theorem: Local form of the iterated logarithm.** *If x is a regular point of the movement at time s, and $X(s) = x$, a. s.*

$$\overline{\lim_{t\to s}}\frac{|X(t)-X(s)|}{\sqrt{(t-s)\lg_2(t-s)^{-1}}} = \sqrt{2}\sigma(x,s) \tag{22.1}$$

Proof. We first assume $\sigma \neq 0$.

We may suppose x and $s = 0$. Let us define $Z(\theta(\tau))$ as previously. Since $Z(\theta(\tau))$ follows a Brownian process, it results from a theorem of M. Khintchine that

$$\overline{\lim_{\tau\to 0}}\frac{|Z(\theta(\tau))|}{\sqrt{\tau\lg_2\tau^{-1}}} = \sqrt{2} \tag{22.2}$$

But, if τ is sufficiently small, outside of cases of probability $\Xi(\tau)$,

$$Z(\theta(\tau)) - Z(0) = X(\theta(\tau)) - X(0) - \int_0^{\theta(\tau)} a(X(t),t)dt$$

and a. s.

$$\lim_{\tau\to 0}\frac{\theta(\tau)}{\tau} = \frac{1}{\sigma^2}.$$

As

$$\left|\int_0^{\theta(\tau)} a(X(t),t)dt\right| = 0[\theta(\tau)] = 0(\tau)$$

$$\text{a.s.}\qquad \overline{\lim_{\tau\to 0}}\frac{|Z(\theta(\tau))|}{\sqrt{\tau\lg_2\tau^{-1}}} = \overline{\lim_{\tau\to 0}}\frac{|X(\theta(\tau))|}{\sqrt{\tau\lg_2\tau^{-1}}},$$

$\tau = \tau(\theta)$ being the inverse function of $\theta(\tau)$.

$$\text{a.s.}\qquad \overline{\lim_{\tau\to 0}}\frac{|X(\theta(\tau))|}{\sqrt{\tau(\theta)\lg_2\tau(\theta)^{-1}}} = \overline{\lim_{\theta\to 0}}\frac{|X(\theta)|}{\sqrt{\sigma^2\theta\lg_2\theta^{-1}}}.$$

Relaxing the hypothesis $X(s) = 0$, $s = 0$, (22.2) becomes (22.1), qed.

Let us suppose that $\sigma = 0$. Define

$$Z'(t) = Z(t) + Y(t),$$

with $Y(t)$ a random function, with independent increments, independent of $Z(t)$, and $\Delta Y(t)/\sqrt{\Delta t}$ following a symmetric Gaussian distribution with standard deviation ε.

Let us now denote by $\theta(\tau)$ the random time at which:

$$\int_0^{\theta(\tau)} \sigma^2 \left(X(s), s\right) ds + \varepsilon^2 \theta\left(\tau\right) = \tau$$

resp. if $|X(u)| = 1$ for the first time at time u_1, let $\theta\left(\tau\right)$ denote the time at which

$$\int_0^{u_1} \sigma^2 \left(X(s), s\right) ds + \overline{\sigma}^2 \left(\theta\left(\tau\right) - u_1\right) + \varepsilon^2 \theta\left(\tau\right) = \tau \ .$$

If M^2 denotes the maximum of $\sigma^2 + \overline{\sigma}^2 + \varepsilon^2$, for $|x| \leq 1$, $\varepsilon < 1$, then one has:

$$\frac{1}{\varepsilon^2} > \frac{\theta\left(\tau'\right) - \theta\left(\tau\right)}{\tau' - \tau} > \frac{1}{M^2} \ .$$

The function $Z'\left[\theta\left(\tau\right)\right]$ is still a Gaussian process with independent increments, and one has:

$$\overline{\lim_{t\to 0}} \left| \frac{Z'(t) - Z'(0)}{\sqrt{t \lg_2 t^{-1}}} \right| = \sqrt{2}\varepsilon \ .$$

If $\overline{\lim_{t\to 0}} X(t)/\sqrt{t \lg_2 t^{-1}}$ were $> \alpha > 0$ with probability $> \beta > 0$, there would exist, with this probability an instant t very small such that

$$X(t)/\sqrt{t \lg_2 t^{-1}} > \frac{\alpha}{2} \ .$$

Since the probability law of $Y\left(t\right)$ is symmetric, the probability such that at the same instant one would have:

$$\left| \frac{Z'(t) - Z'(0)}{\sqrt{t \lg_2 t^{-1}}} \right| > \frac{\alpha}{2}$$

is $> 1/2$. Hence, one would have:

$$\Pr \left\{ \overline{\lim_{t\to 0}} \left| \frac{Z'(t) - Z'(0)}{\sqrt{t \lg_2 t^{-1}}} \right| > \frac{\alpha}{2} \right\} > \frac{\beta}{2},$$

which implies $\beta = 0$. The theorem follows in the case $\sigma = 0$, qed.

...

XV. *Changes of variables.* Let $\varphi\left(x, t\right)$ be an increasing function of x continuous with respect to (x, t). Define $Y\left(t\right) = \varphi\left(X\left(t\right), t\right)$. Let $G\left(x, y, s, t\right)$ be the probability that $Y\left(t\right) < y$, if at time s, one had $Y\left(s\right) = x$. $G\left(x, y, s, t\right)$ is equal to the probability that $X\left(t\right) < y'$ when one had: $X\left(s\right) = x'$.

It is easily verified that $G\left(x, y, s, t\right)$ satisfies Chapman's equation, for x varying between $\varphi(-\infty, s)$ and $\varphi(+\infty, s)$. But, in order that G satisfies Kolmogorov's equation, it is necessary to make some further hypothesis on φ. We assume that φ'_x, φ''_{x^2} and φ'_t exist, and that φ'_x and φ''_{x^2} are continuous with respect to t and x.

We apply the finite increments formula:

$$\begin{aligned} Y(t+\Delta) - Y(t) &= \varphi[X(t+\Delta), t+\Delta] - \varphi[X(t), t+\Delta] \\ &\quad + \varphi[X(t), t+\Delta] - \varphi[X(t), t] \\ &= \varphi'_x[X(t), t+\Delta](X(t+\Delta) - X(t)) \\ &\quad + \frac{1}{2}\varphi''_{x^2}\left[X', t+\Delta\right](X(t+\Delta) - X(t))^2 \\ &\quad + \varphi[X(t), t+\Delta] - \varphi[X(t), t] \ , \end{aligned}$$

for X' between $X(t)$ and $X(t+\Delta)$. Consequently:

$$\int_{|y-x|>\varepsilon} dG(x,y,s,t) = o(\Delta)$$

uniformly in every region of the plane (y,t) which is the image by the transformation $y = \varphi(x,t)$ of a region of the plane (x,t) which does not contain singular points and in which φ'_x and φ''_{x^2} are bounded.

Likewise

$$\begin{aligned} A &= \frac{1}{\Delta}\int_{|z-Y|<\varepsilon} [z - Y(t)] d_z G(Y(t), z, t, t+\Delta) \\ &= \varphi'_x[X(t), t]\, a(X(t), t) + \varphi'_t[X(t), t] \\ &\quad + \frac{1}{2}\varphi''_{x^2}[X(t), t]\, \sigma^2(X(t), t) + \Xi(\Delta) \ , \\ \overline{\sigma}^2 &= \frac{1}{\Delta}\int_{|z-Y|<\varepsilon} [z - Y(t)]^2 d_z G(Y(t), z, t, t+\Delta) \\ &= \varphi''_{x^2}[X(t), t]\, \sigma^2(X(t), t) + \Xi(\Delta) \ . \end{aligned}$$

The expressions $\Xi(\Delta)$ in the preceding formulae converge to zero uniformly in every region which does not contain singular points in which φ'_x, φ''_{x^2} and φ'_t are bounded, and φ'_x and φ''_{x^2} are continuous with respect to t.

An important particular case. Let

$$Y(t) = \varphi(X(t), t) = \int_0^{X(t)} \frac{dx}{\sigma(x,t)} \ .$$

Assume that σ'_t exists as well as σ'_x, and that σ'_x is continuous with respect to t. One then has

$$A = \frac{a(x,t)}{\sigma} - \int_0^x \frac{\sigma'_t}{\sigma^2}\, dx - \frac{1}{2}\sigma'_x \quad ,$$

$$\overline{\sigma}^2 = 1 \ .$$

Remark. One might consider more complicated changes of variables by introducing instead of t and x functions $\varphi(t,x)$, etc. But we shall not need these.

. . .

XXI. *Assume that* a, σ *and* $1/\sigma$ *are continuous, for* $\alpha \le x \le \beta, \tau_1 \le s \le \tau_2$. *Consider two particles whose random movement obeys the same law* $\{F(x,y,s,t)\}$, *the two particles moving independently from each other. Let* $X_1(t)$ *and* $X_2(t)$ *be* *the positions of the particles at time* t*. If one has* $\alpha < \alpha' < X_1(\tau_1) < X_2(\tau_2) < \beta' < \beta$*, then the probability that the two particles meet before time* τ_2 *tends to 1 if* $X_1(\tau_1) - X_2(\tau_1) \to 0$.

Proof. Suppose $X_1(\tau_1) < X_2(\tau_1)$, and let $Z(t) = X_1(t) - X_2(t)$. If, at some time t, one has $Z(t) > 0$, it implies that the two particles have met before t. The probability distribution of $Z(t) - Z(\tau_1)$ is, if $(t - \tau_1)$ is very small, and $\alpha' < X_1(\tau_1) < X_2(\tau_1) < \beta'$, $X_2(\tau_1) - X_1(\tau_1) < \varepsilon$, very close to a Gaussian distribution. One then proves XXI in a quite similar manner to V, and one verifies that the probability for the particles to have met before time $\tau > \tau_1$ tends to 1 uniformly with respect to $X_1(\tau_1)$ and $X_2(\tau_1)$ $\left(\alpha' < X_1(\tau_1) < X_2(\tau_1) < \beta'\right)$ if $X_2(\tau_1) - X_1(\tau_1) \to 0$. In case one may take α and $\beta = 0$, a different method allows to prove that the probability in question is of the form $1 + 0(X_1(\tau_1) - X_2(\tau_1))$, which probably extends to the general case.

. . .

XXIII. **Theorem.** *If* $F(x, y, s, t)$ *is continuous with respect to* x*, for every* y*, whatever* (s, t) *in* $[S, T]$ *, and if* $X(t)$ *is a. s. continuous for* $S \leq t \leq T$ *for every value of* $X(s)$*, then* $F(x, y, s, t)$ *is monotone with respect to* x*.*

Proof. We consider again two moving particles whose random movement obeys the same law $F(x, y, s, t)$, the two particles moving independently from each other. Let $X_1(t)$ and $X_2(t)$ be the positions of the two particles at time t. Assume $X_1(s) < X_2(s)$. The probability $F(x, y - 0, s, t)$ is the probability that $X_1(t)$ be $< y$, $X_1(s)$ being $= x$, and $F(X_2(s), y - 0, s, t)$ is the probability that $X_2(t) < y$. $X(u)$ being a. s. continuous, three cases only are possible (apart from movements with zero total probability).

1st case: one has $X_1(u) \leq X_2(u)$ for all u between s and t;
2nd case: one has $X_1(u) = X_2(u)$ for (at least) one $u < t$;
3rd case: one has $X_1(u) < X_2(u)$ for all $u < t$, but $X_1(t) = X_2(t)$.

In the first and second cases, if $X_2(u)$ is $< y$, $X_1(u)$ is a fortiori $< y$. The probability that $X_i(u) < y$ in the case where the curves $X_1(u)$ and $X_2(u)$ meet is – taking into account the fact that the probability such that $\max_{s \leq u \leq t} |X_1(u)| > K$ or $\max_{s \leq u \leq t} |X_2(u)| > K$ goes to 0 if $K \to \infty$, and that by hypothesis $F(x, y, s, t)$ is continuous with respect to x - the same for $X_1(u)$ and $X_2(u)$.

It follows that

$$F(x, y - 0, s, t)$$

is monotone with respect to x. The same is true for $F(x, y + 0, s, t)$, and by hypothesis

$$F(x, y, s, t) = F(x, y + 0, s, t),$$

which ends the proof of the theorem.

5 Reading notes

5.1 Kolmogorov's equation and the conditions (1), (2), (3), (4), (5)

The theory of Kolmogorov's equation begins with the fundamental paper (1931) by Kolmogorov, the sources of which are fairly well known: the mathematical theory of Markov chains, published from 1907 onwards by Markov, and whose aim is to extend the limit theorems of probability theory to cases where the independence hypothesis does not necessarily hold, met a rather cold welcome from the international mathematical community. It was taught by S. Bernstein in Kharkov during the first world war, but one cannot find any other echo from this beautiful contribution from Markov, which does not even acquire a name; indeed, the term "Markov chain" dates from 1929 and this is no coincidence. (During the International Congress of Mathematicians in Bologna (1928), when Hadamard and Hostinský presented their works about the ergodic principle in the framework of card shuffling, as initiated by Poincaré, Pólya informed them about the previous works of Markov concerning chains of events: the "Markov chains" had been baptized!)

On the other hand, since the very beginning of the century, and independently from Markov's mathematical works, theoretical physicists had developed Markovian-type computations which, they believed, were likely to explain all sorts of diffusion phenomena, or even all physical phenomena as soon as one relaxes the analytical determinism to replace it by a determinism of another kind, which does not fix the evolution of the system from the present to the immediate future, but yields a (well-determined) probability distribution for the immediate future, if the present is given, (Einstein 1905; Chapman 1928; Ornstein-Uhlenbeck 1930; Schrödinger 1932, 1946; Nelson 1967 etc.). For diverse reasons, many other scientists have been interested in discrete or continuous Markovian schemes, Bachelier being one of the first, but also mathematical actuaries, namely those in the Scandinavian school (Cramér 1930), in order to describe the evolution of a client's account, or the mathematical biologists who created the genetics of populations (Fisher 1930), or again the telecommunication engineers (Brockmeyer et al 1948). But, at the end of the twenties, Markov's theory began to interest mathematicians again, independently from any idea of applications. The mathematician analysts of the new generation, belonging to the Moscow and Paris Schools in particular, saw there a sort of natural extension of the theory of functions studied in the Paris Baire-Borel-Lebesgue School, where one is interested in the "arbitrary functions", those which, following Dirichlet, associate to one value of the variable x, a "well determined" value of the function $f(x)$.

Since the fairly mysterious idea of choosing a number at random took a precise analytic meaning in the setup of the new theory of functions, it became possible to get interested in functions obeying certain laws of ("determinations" in) probability, instead of satisfying "analytical expressions" as Euler or Lagrange phrased it. Indeed, as early as 1905, Borel suggested replacing the time-honored "geometrical probability" by the Borel measure associated with the Lebesgue integral, and, quite soon afterwards, Paul Lévy and Richard von Mises (independently) defined a “probability law” in a general finite dimensional euclidian space as a positive measure with unit mass in the sense of Borel-Lebesgue-Vitali-Fubini-Young-Riesz-Hausdorff-Radon-Carathéodory-Hahn-etc. This might have been a starting point for the mathematical construction of a theory of functions defined at random: a differential and integral stochastic calculus, a stochastic Fourier theory, in short a stochastic analysis, of which the well advanced study of Markov chains would give a first sketch in the discrete variable case.

The continuous variable case presented, of course, more difficulties, but it had an almost obvious interest. Already, Bachelier, as he studied quotations in the Bourse of Paris, and adapted the methods of the classical theory of the gamblers’ ruin, introduced, in his way, the general diffusions, homogeneous in space. In particular, he showed the link between these new “continuous probabilities” and the theory of heat: when all is fair (one might say in equilibrium) and continuous, probability diffuses as heat, (Taqqu 2001). Wiener, on his side, and this time in a strictly mathematical mode, remaining inside the new theory of functions, constructed, around 1923, the law of probability of diffusions which are homogeneous in space and time, which are now called “Brownian motions”. His first construction, too complicated to be usable, was improved at the beginning of the thirties by Wiener himself, Paley and the Polish School, Steinhaus, Marcinkiewicz, Zygmund, Kac,...The mathematical theory of Brownian motion took its shape in a rigorous analytical framework, including Lebesgue’s measure, “independent functions”, Wiener’s measure, which constitute a well identified set of well known mathematical objects (Kahane 1997, 1998).

Kolmogorov’s paper (1931), inspired by the works of Bachelier about homogeneous diffusions, and by those of Hostinský-Hadamard on Markov chains, aims at defining a unified analytical set up for all “stochastically defined processes”. The objective is to study the probabilistic Markovian schemes, as generally as possible, in dimension one, (then in higher dimension in (1933a)), in the discrete and continuous cases, i. e. the systems of “well-defined” probabilities which satisfy Chapman’s equation – whose probabilistic interpretation and analytical nature are clear. Kolmogorov develops in particular a remarkable study of continuous time-space which we very briefly discuss below. This article and its companion (1933a) mark the birth of the mathematical theory of diffusions. All immediately posterior works by Bernstein, Khinchin, Petrowski, Feller, Kolmogorov and of course Doeblin, which we shall now discuss, are direct descendants of the former, one way or another: it would also be of interest to study in detail the further developments of Kolmogorov’s problem, which indeed motivated an important part of the theory of probability in the second half of the XXth century. However, describing the fundamental works of Itô, Doob, those of the Russian, American, Japanese, French,... schools is an unreasonable task, and we shall simply refer the reader to the main introductions of (Ikeda-Watanabe 1981, 1996; Stroock-Varadhan 1979, 1987; Dynkin 1965; Kendall 1990; Shiryaev 1989; Ventsel 1994), and, with the advent of the new millenium (Meyer 2000; Varadhan 2001; Watanabe 2001, etc.).

5.1.1 The local conditions (1), (2), and Kolmogorov's theory

The local conditions (1) and (2) in the Pli had been introduced by Kolmogorov in his second memoir (1933a), they already appeared in Kolmogorov's 1931 memoir in a global form. We shall follow the 1933 text of Kolmogorov, allowing ourselves to modify very slightly the notation in order to bring them closer to Doeblin's; we briefly detail the original computations of Kolmogorov with the help of Doeblin's movement X, which Kolmogorov has clearly in mind, but without writing it down in 1931–1933. It may be useful to recall that Kolmogorov's axioms date from 1933, (1933b) and that, without it, the letter X has an intermediary status, somewhere between a physical concept, a financial or actuarial metaphor, or again some kind of mathematical object, whose properties may be clear for a number of scientists among whom one should of course find Kolmogorov himself, but which are resolutely vague, or empty, for the great majority of them.

In 1931 as well as in 1933, Kolmogorov aimed at deriving from Chapman's equation, under well defined mathematical conditions, the (so-called Kolmogorov!) parabolic equations, for which he hopes to find probabilistic solutions and their behavior at infinity as in the case of chains.

Thus, let $F(x,y,s,t) = \Pr\{X_t < y/X_s = x\}$, where X is a continuous movement in the sense of Doeblin, F being the only object featured in the computations and theory under study by Kolmogorov. We assume, as does Kolmogorov, that F has a density $f(x,y,s,t)$ with respect to y, which is as many times differentiable as one desires. The functions F, as well as the functions f, are interrelated by Chapman's equation:

$$f(x,y,s,t) = \int f(x,z,s,s+\Delta) f(z,y,s+\Delta,t)\, dz\,.$$

Let us assume that Δ is small, and let us see how this equation allows us to recover Kolmogorov's PDE equation, under some reasonable conditions, (1933a, §1).

(As Kolmogorov and Doeblin), we let ourselves be guided by the behavior of the movement X in the neighborhood of s. Since the movement is continuous, the only "probable" values z of X are near x. So, let U be a neighborhood of x, and write:

$$\begin{aligned} f(x,y,s,t) = &\int f(x,z,s,s+\Delta) f(x,y,s+\Delta,t)\, dz \\ &+ \int_U f(x,z,s,s+\Delta)\{f(z,y,s+\Delta,t) - f(x,y,s+\Delta,t)\}\, dz \qquad \text{(a)} \\ &+ \int_{\mathbb{R}-U} f(x,z,s,s+\Delta)\{f(z,y,s+\Delta,t) - f(x,y,s+\Delta,t)\}\, dz \end{aligned}$$

The first integral equals $f(x,y,s+\Delta,t)$. In order to estimate the second integral, it suffices to develop the term between { } with Taylor's formula, written up to order 2:

$$
\begin{aligned}
& f(z, y, s+\Delta, t) - f(x, y, s+\Delta, t) \\
& \quad = (z-x)\,\frac{\partial f}{\partial x}(x, y, s+\Delta, t) + \frac{1}{2}(z-x)^2\,\frac{\partial^2 f}{\partial x^2}(x, y, s+\Delta, t) \qquad \text{(b)} \\
& \qquad + o\,(z-x)^2
\end{aligned}
$$

that is, integrating on the set U, and under conditions on f which imply conditions 1 and 2 (except in certain points called singular by Doeblin), one finds that the second integral is equal to

$$
\begin{aligned}
& \left[a\,(x,s)\,\frac{\partial f}{\partial x}(x,y,s,t) + \frac{1}{2}\sigma^2(x,s)\,\frac{\partial^2 f}{\partial x^2}(x,y,s,t)\right]\Delta \qquad \text{(c)} \\
& \quad + \int_U o\,(z-x)^2 f\,(x,z,s,s+\Delta)\,dz\ .
\end{aligned}
$$

Then, Kolmogorov imposes a “Lindeberg condition” on X, which is sufficiently strong so that the integral part of this last expression and the last integral in the preceding sum are $o\,(\Delta)$. Therefore, we have established the first equation of Kolmogorov: f as a function of x and s solves the equation:

$$
L(u)\,(x,s) = \frac{\partial u}{\partial s}(x,s) + a\,(x,s)\,\frac{\partial u}{\partial x}(x,s) + \frac{1}{2}\sigma^2(x,s)\,\frac{\partial^2 u}{\partial x^2}(x,s) = 0\ . \qquad \text{(d)}
$$

Likewise, Kolmogorov shows that f as a function of y and t satisfies the adjoint, so called Fokker-Planck equation, as it has been obtained (without precise mathematical conditions) in statistical physics a few years earlier.

At this point it suffices to look for some adequate probabilistic solutions of the parabolic equation $L\,(u) = 0$, and to study them, which is precisely what Kolmogorov does in a number of particular cases [1931, 1933a], whilst asking the following questions (1931, p. 452):

1) *Unter welchen Bedingungen existiert eine solche Lösung der Gleichung* (133) [which refers to Fokker-Planck equation]?

2) *Unter welchen Bedingungen kann man behaupten, dass diese Lösung wirklich den Gleichungen* (85)*und* (86) [Chapman’s equation for the probability density f] *genügt?*

Such is “Kolmogorov’s problem”, which Doeblin studies in the Pli.

5.1.2 On condition (3) and Feller’s memoir

In 1936, Feller replaced the Lindeberg-Kolmogorov condition by condition (3) of the Pli. This (Feller) condition plays a very important role in the theory: under conditions (1), (2), (3) and some adequate conditions of analyticity, Feller showed that F solves the equation $L\,(u) = 0$, as a function of x and s, and that f solves the adjoint equation as a function of y and t. He then proved a very general theorem of existence and unicity in the set-up of the Hadamard-Gevrey theory. Indeed, the theory of parabolic equations developed considerably at the beginning of the XXth century, by S. Bernstein, E. Holmgren, E. E. Levi, J. Hadamard, M. Gevrey, among others, renewed the historical works of Laplace, Fourier and Dirichlet. In particular, they considered anew the study of existence and uniqueness and the difficult boundary problems. The famous Goettingen School immediately incorporated these questions into its program, and it is conceivable that Feller, a brilliant representative from the Goettingen School, was able to go further than Kolmogorov's works on Kolmogorov's equation [14].

One of the aims of Doeblin in his works on Kolmogorov's equation was to establish a theorem of existence *without* the strong analytical conditions of Feller. He proceeds using approximation in distribution from Feller's case, using in particular a result on uniform continuity on F, theorem XXII.

The celebrated Feller memoir (1936), which was the starting point of Fortet and of many others, is strictly analytical in nature – nothing stochastic appears explicitly, whereas, on the other hand, condition (3) is clearly a strong continuity condition on the movement: the present being given, variations of amplitude bigger than η during a time interval of length Δ have probability $o\,(\Delta)$. It is quite clear that Feller, as well as Kolmogorov, thought first in terms of the movement X, but they computed the law f, and wrote analytical theorems. There is no indication, or published version, of any construction by Kolmogorov and Feller of a continuous version of their movements which would allow them to discuss its trajectories, and to deduce results from this approach, even after 1933, when Kolmogorov's axioms got published and the theory of random functions began to develop.

5.1.3 Bernstein's theory and Doeblin's conditions (4), (5)

It is time to say a few words about Bernstein, who had an important influence on Doeblin. Serge Bernstein was one of the great mathematicians of the first half of the 20th century – his mathematical production is considerable, and does not need to be recalled here. Although during the first world war, Bernstein got interested in probability theory in order to earn his living (he was an actuary) as well as for pedagogical reasons (he taught probability and statistics to J. Neyman), he soon became fascinated by the mathematical and physical aspects of the theory. As such, he is the author of one of the first (non set-theoretic) axiomatics of probability theory, and he published in the twenties some important papers about asymptotic normality in cases of weak dependence (1926). As early as 1931, he set about Kolmogorov's equation. There, he clearly saw an occasion to construct a probabilistic solution to the parabolic equations, of which he was one the world's best specialists.

[14] Kolmogorov knew very well that the analytic solution of "his" problem was to be found in Göttingen; he had asked for a Rockefeller grant in 1932: "to pursue studies in the field of theory of probability and analysis at the Dept. of Mathematics of Göttingen with Prof. R. Courant". Kolmogorov's fellowship was scheduled to start in May 1, 1933. It was too late, Courant had been dismissed by the Nazis in April 1933. Following the advice of Hermann Weyl, Kolmogorov decided not to go to Göttingen but to Paris with Hadamard. But Kolmogorov never obtained his visa from the Soviet authorities, [Siegmund-Schultze 2001, p. 132].

Bernstein might have started from one of the very rare, non-analytic commentaries of Kolmogorov (1931, p. 448). After obtaining the equation $L(u) = 0$, and having shown the important role played by the conditions (1) and (2) related to a and σ, Kolmogorov added (we still keep Doeblin's notation): "The true meaning of a and σ is the following: $a(x, s)$ is the mean speed of the variation of the parameter x during an infinitely small time interval, while $\sigma(x, s)$ is the differential dispersion of the process. The dispersion of the difference $(y - x)$ during the time interval Δ is

$$\sigma(x, s)\sqrt{\Delta} + o\left(\sqrt{\Delta}\right) = 0\left(\sqrt{\Delta}\right) ,$$

while the mean of this difference is:

$$a(x, s)\,\Delta + o(\Delta) = 0(\Delta) \text{."}$$

It was tempting to build a theory of stochastic differential equations, using this commentary as a foundation. As early as 1932–1933, Bernstein considered stochastic difference equations of the form:

$$\Delta y_i = a(y_i, t_i, \alpha_i)\,\Delta t_i + \sigma(y_i, t_i, \alpha_i)\sqrt{\Delta t_i}$$

the α_i's indicating that a random choice is being performed at time t_i, independently of the past, once the present is given.

It then suffices to study the asymptotic behavior of the laws of the solutions of these equations to obtain, under some adequate conditions, a probabilistic solution to the Fokker-Planck equation associated to the mean values of the random data a and σ (Bernstein 1938, p. 11).

Bernstein's works had been published in French by Doeblin in one of the *fascicules* of the Colloque of probability theory which was held in Geneva in October 1937, and was the first international congress entirely devoted to the theory of probability and its applications. These works do not seem to have impressed Doeblin very much, as he thought the conditions of Bernstein far too restrictive; however, these works contain a study of the infinite branches of the movements which is quite original and which, almost certainly, contributed to the construction by Doeblin of his theory of regular movements.

Bernstein examined the particular case of the equation

$$\Delta y = y^2 \Delta t + \alpha\sqrt{\Delta t}$$

in which α takes the values $+1$ or -1 with probability 1/2, (Bernstein 1938, pp. 6–9). He then observed that, starting from 0 at time 0, and letting Δt converge to 0, the probability that y is infinite at time $t = 6$ is greater than 0.0061. Thus, with strictly positive probability, there has been "explosion" (for a given time) following the terminology used in the fifties. To prevent this phenomenon, one needs to impose that $a(x, s)$ grows at infinity at most as x; this is the condition of "quasi-linearity" of Bernstein, which is found later in the "classics" (McKean 1969, p. 66, problem 2; Ikeda-Watanabe 1981, Chap. IV, Theorem 2.4, for example).

One may understand why Doeblin added to the conditions (1), (2), (3), the conditions at infinity (4), (5) which allow (random) explosions which cannot happen too brutally, so that after a change of scale in x one still can construct by interpolation a continuous version of X. The contemporaries of Doeblin, notably Feller and Fortet, limit themselves to bounded data. The conditions (4) and (5) allow one to go beyond these limitations. We have not found whether they have any posterity.

5.2 Doeblin's theory of regular movements, his representation theorem

Doeblin's theory of regular movements is deliberately a pathwise one. Let X be a continuous movement, defined up to a change of scale, whose law satisfies the conditions (1) to (5); there are many of them, at least under Feller's hypotheses (1936). Then Doeblin proves that any such movement is locally Gaussian, and hence possesses a number of the regularity properties of Brownian motion. We discuss this point in detail below. Before doing so, let us observe that Doeblin is one of the very first authors to seriously treat Kolmogorov's problem using this new "point of view", and is the first who goes so far in this direction. His results will be only rediscovered fifteen or twenty years later, and some of them still do not have any equivalent. The only comparable study, although made a little later, is due to Robert Fortet (1941, 1943) who treats the case $\sigma = 1$ and a bounded satisfying Feller's conditions. From the start, Fortet, like Doeblin shows the existence of a continuous version of his movements and works on the space of continuous functions. He stresses in his 1943 memoir that this is indeed a "new point of view" (1943, Chap. II) which allows him to compute in all rigor what he calls, following Bernstein, the absorption probabilities, whose links with the boundary problems of parabolic equations had been indicated by Bernstein himself in his plenary Zürich paper (1932) and which Doeblin treats, in his framework, in §XVII of the Pli. Some similarities between the memoirs of Fortet and Doeblin are noteworthy, and may be partly explained by the fact that both actively participated in the "Séminaire Borel" at IHP, which was devoted in 1937–1938 to the theory of random functions. Lamperti (1966, 1977) writes that Fortet's memoir opened a new era in diffusion theory. Doeblin certainly belongs to this era, since 1938 at least, by anticipation. A comparison of the texts of the two authors shows that although their point of view is the same, their methods are different and have little intersection, Fortet remaining more analytically inclined than Doeblin. Apart from this same new viewpoint, the most remarkable resemblance between the memoirs of Doeblin and Fortet lies in the fact that the two authors prove, each of them in his own set-up, a "representation theorem" of the solutions of Kolmogorov's problem starting from the Brownian motion of Wiener-Bachelier, which clearly manifests the mathematical strength of their viewpoint. This was followed by Itô's representation theorem (1946) which then played a determining role.

From the pathwise viewpoint, the idea of associating with a diffusion X a "compensated" process Z which follows the trajectories of a standard Brownian motion is a natural one. It is (consequently) absent from the great memoirs of Khinchin, Kolmogorov and Feller, but the idea of compensation is present in the works of Lévy on additive processes (1934, 1937, Chap. VII] and mostly (1948, Chap. III, no. 17, 2°, p. 72), as well as in the works of Lévy and Doeblin on the sums of random variables, and even, between the lines, in the seminal paper of Kolmogorov (1931), for example in his solution of the (Lévy) "Bachelier case", §16, p. 453.

However, Doeblin's method goes much further and the change of time which he adopts seems to be original. It is usually attributed to (Volkonskii 1958;, see e.g. Dynkin 1965; Williams 1979). In any case, there does not seem to be much use for random time-changes in the study of diffusions before the end of the fifties. Another comparable example, although coming later, was proposed by Lévy towards 1943 to prove the conformal invariance of the planar Brownian curve (Lévy 1948, Théorème 56.1, this example being analysed in B. Locker's thesis 2001). It is of course well known that Bachelier and Lévy use stopping times freely in their fine study of linear Brownian motion, see (Lévy 1939, 1948; Chung 1989), but these random times allowed them mainly to obtain decomposition formulae via the strong Markov property, which does not correspond to the use made here by Doeblin.

Lemma IX is a version of the Dubins-Schwarz (1965) – Dambis (1965) theorem. The only related result at that time was Lévy's characterization of Brownian motion, (1937, no. 52), which was then relative to the continuous processes with independent increments and not to martingales, see Loève, (1955/1977), vol II pp. 210–212 where one finds the full Lévy theorem, first stated in Lévy (1948), thm. 18.6. Doeblin's proof relies again upon a central limit theorem for weakly dependent variables, which is due to Bernstein (1926).

The notion of (positive) martingale and its denomination (which Doeblin does not use) are due to Ville, in his 1939 thesis. It was considered by Lévy under another name since 1934 (e.g., 1937, no. 65); in fact Lévy called the martingale property "condition C"). Hence, "martingale methods" were part of the probabilistic working kit at the end of the thirties, at least in Paris, and Doeblin knew, of course, the works of Lévy and Ville on this topic. Nonetheless, the use which Doeblin makes here of the martingale property seems to be original; it is, at the very least, remarkable.

The systematic study of martingales really began with Doob's fundamental paper (1940) which establishes the convergence theorem and with his 1953 book. From then on, it played a central role in the theory of probability of the second half of the XXth century. Let us recall that Jean Ville (1910–1989) was, with Doeblin, the main organizer of the séminaire Borel of probability, founded in 1937–1938 in IHP. This séminaire was clearly at the origin of some of the themes developed in the Pli. After the war, Ville left the University and martingale theory to embrace a career of scientific consultant for telecommunications. In 1956, he became Econometrics Professor at the Sorbonne, while continuing his consultancy activities. Ville, as well as Fortet, kept a luminous memory of Wolfgang Doeblin; his obvious superiority, his way of understanding mathematics, which was so intuitive, and at the same time his astonishing technical virtuosity, impressed them both very much, as was also the case with Lévy and Loève.

5.3 The change of variables formula in the theory of diffusions

In his 1931 paper, Kolmogorov devoted the 17th section, entitled “Eine Transformation”, to changes of variables in space and time in Kolmogorov’s equation. The aim of such formulae is to make precise how the coefficients a and σ are transformed when, for example, x is changed in $\varphi(x, t)$. Kolmogorov showed in a particular case how it is possible to reduce his equation to the heat equation. Feller (1936) developed the same idea (which, in fact, is classical in the theory of parabolic equations) and he introduced the “important particular case” considered by Doeblin.

Doeblin’s new point of view consists in working directly on the movement $X(t)$ which is being changed in $Y(t) = \varphi(X(t), t)$. It is then natural to express the increment of Y in term of those of X and t, with the help of the formula of finite increments. When φ is regular enough

$$\Delta Y = \varphi'_x \Delta X + \frac{1}{2}\varphi''_{xx} (\Delta X)^2 + \varphi'_t \Delta t$$

From there, Doeblin deduces (here, we assume that Y is integrable, to obtain a simpler formula):

$$E\left(\Delta Y / X(t) = x\right) = L(\varphi)(x, t)\,\Delta t + o(\Delta t)$$

and, when Y is square integrable:

$$E\left((\Delta Y)^2 / X(t) = x\right) = \left(\sigma\varphi'_x\right)^2 (x, t)\,\Delta t + o(\Delta t)$$

These formulae summarize all the stochastic information contained in Kolmogorov’s equation (see e.g., Stroock 1987, pp. 25–26).

Clearly, Doeblin’s formula of finite increments is a sort of Itô’s formula without Itô’s integral, in the context of the theory of “regular movements”. Should therefore Doeblin be credited as the inventor of this formula, or at least be recognized as a pioneer?

In fact, this situation is very commonly encountered in the history of Mathematics. It suffices to think about Taylor’s formula, or the formula of changes of variables in multiple integrals, or Stokes formula, and what about the “Heine-Borel” theorem? Who is the pioneer of what? It appears very quickly that this question does not have a great meaning or importance, even if for the different persons concerned it is of great interest, and may be the starting point of involved arguments. What is truly important is to understand the moment, the place, the occasion when a theory and/or the problems which nourish it, necessitates the introduction of a formula and/or a new computation, even calculus, as if the problems contained in themselves these key formulae, and the scientists devoted to solving these problems found and experienced these formulae in the middle of their investigations, although at first they were not particularly impressed with the radical originality of these important formulae. Itô himself explains in (1951b) how his formula appears surreptitiously in his first works of 1942–1944 about Kolmogorov’s problem, without his noticing its importance and novelty, and that he published the formula only in his later works (1950, 1951b), after practising it sufficiently.

It is most likely that the same formula has been used in different forms, independently even from Itô's theory, as soon as Kolmogorov's problem was clearly posed, and attempts were made to solve it using a stochastic approach, in a more or less clear way. Doeblin did exactly this in 1938–1940, but was he the first?

Let us look again at the princeps paper of Kolmogorov (1931), and let us follow his proof of the second fundamental equation (that of Fokker-Planck).

Let $R(x)$ a function of the variable x assumed to be regular and 0 at infinity. Kolmogorov's computation, translated into the Doeblin set-up, may be written as follows (1931, p. 449):
first, observe that:

$$(+) \qquad E\left(\Delta R(X(t))/X(t)=y\right) = L(R)(y,t)\,\Delta t + o(\Delta t),$$

where:

$$L(R)(y,t) = a(y,t)R'(y) + \frac{1}{2}\sigma^2(y,t)R''(y) \quad .$$

It then suffices to integrate the equality (+) with respect to the probability $f(x,y,s,t)\,dy$, and then integrate by parts to obtain:

$$\begin{aligned}\frac{1}{\Delta t}E\left(\Delta R(X(t))/X(s)=x\right) &= -\int \frac{\partial}{\partial y}\left[a(y,t)f(x,y,s,t)\right]R(y)\,dy \\ &\quad +\frac{\partial^2}{\partial y^2}\left[\sigma^2(y,t)f(x,y,s,t)\right]R(y)\,dy + o(1)\end{aligned}$$

which is, since R is a test function, the Fokker-Planck's equation:

$$\begin{aligned}\frac{\partial}{\partial t}f(x,y,s,t) &= -\frac{\partial}{\partial y}\left[a(y,t)f(x,y,s,t)\right] \\ &\quad +\frac{\partial^2}{\partial y^2}\left[\sigma^2(y,t)f(x,y,s,t)\right] .\end{aligned}$$

In 1932, in his Zürich paper (delivered in his absence), Bernstein showed how a similar method, which also hinges, as in Doeblin's work, upon the equality of finite increments, leads to the first Kolmogorov equation for the distribution function $F(x,y,s,t)$ (Bernstein 1932, p. 300).

Other examples may be found in Lévy's writings during the war and in his treatise (1948, e.g., Chap. III, §16; see Locker 2001).

A. Shiryaev told us that, one day, he asked Kolmogorov how he had been able to derive his equations without being aware of Itô's formula. Kolmogorov smiled and advised Shiryaev to read his 1931 memoir closely. This led Shiryaev to call the change of variables formula (at least for one-dimensional diffusions) the Kolmogorov-Itô formula (Shiryaev 2000, p. 263).

As is well known, the next step in what we might call, following the above discussion, the Kolmogorov-Doeblin formula, is Itô's construction of stochastic integrals, and of diffusions as solutions of stochastic differential equations. Indeed, Itô, since 1942, integrates directly the equation of the movement:

$$dX(t) = a(X(t), t)\, dt + \sigma(X(t), t)\, dB(t).$$

If a and σ satisfy a Lipschitz condition and are quasi-linear in the sense of Bernstein, one can define a process $X(t)$, a. s. continuous, via the formula

$$X(t) = X(0) + \int_0^t a(X(s), s)\, ds + \int_0^t \sigma(X(s), s) dB(s)$$

Under some additionnal regularity conditions bearing on a and σ, it is shown that the law of X satisfies Kolmogorov's equation (e.g. McKean 1969, Chap. 3). Hence Kolmogorov's problem is explicitly solved in this case, and this was one of the main motivations of Itô, as of Bernstein, ..., Doeblin. The change of variables formula now takes the (complete) Itô form:

$$\begin{aligned} dY(t) &= \varphi'_x(X(t), t)\, dX(t) + \frac{1}{2}\varphi''_{x^2}(X(t), t)\,(dX(t))^2 + \varphi'_t(X(t), t)\, dt \\ &= L(\varphi)(X(t), t)\, dt + \left(\sigma\varphi'_x\right)(X(t), t)\, dB(t) \end{aligned}$$

In particular, if $(X(t))$ is the standard Brownian motion:

$$dY(t) = \left(\varphi'_t + \frac{1}{2}\varphi''_{x^2}\right)(X(t), t)\, dt + \varphi'_x(X(t), t)\, dX(t)$$

This Itô formula should be understood as the infinitesimal element of a Itô integral; Itô integrals are missing in Doeblin's theory, as in the theories of all other preceding authors. It is well known that, in the thirties, Paley and Wiener use a notion of stochastic integral in their works of generalized Fourier analysis (McKean 1969). Lévy, on his side, has developed his own theory of stochastic integrals to compute the law of the area of the planar Brownian curve, starting from 1939 (Locker 2001). However, none of these stochastic integrals is well adapted to Kolmogorov's problem. This is the main import of Itô during the war, and of Gihman slightly after, and differently (see for this topic, e.g., Ikeda-Watanabe 1981).

Of course, one may ask why Doeblin did not develop his own theory of stochastic integrals, even if it is impossible to answer this question. We have learnt from Laurent Schwartz (personal communication) that Doeblin intended to develop such integrals since 1938. However he did not do it. Perhaps, he estimated that it would not allow him to solve Kolmogorov's problem in all its generality. In 1938, Bernstein's theory, which is based on stochastic difference equations, and Feller's theory, which is based on the parabolic PDE, already solved Kolmogorov's problem under conditions which were quite comparable to Itô's. To go further, one had to abandon the idea of a simple global representation of the movements. It is plausible that Doeblin, who was meditating on these themes for the previous four years, and whose ability to construct proofs was exceptional, understood that a general theory of the stochastic integral would not be yet able to solve the problem he was considering. In fact, it took about 25 years for Itô's theory to be really accepted and used: McKean's book (1969) was the first book to present a systematic overview of Itô's stochastic integration. Then, Itô's theory proved its versatility and efficiency, in connection with

Doob's martingale theory, so that, enriched by the works of a large number of scientists during the fifties and the sixties (e.g., Meyer 2000), Itô's formula would become the cornerstone, from the seventies onwards, of a myriad of applications of stochastic calculus, of which Wolfgang Doeblin, in his memoir "Sur l'équation de Kolmogorov" was one of the heralds.

5.4 Coupling: some properties of continuity and monotonicity of F

In his Sect. XXI, Doeblin introduces, in the framework of his "regular movements", an original technique of coupling, which he had already experienced in his theory of Markov chains, namely: to show the ergodic principle of Markov chains, according to which the final distribution does not depend on initial conditions, Doeblin considers two independent samples of the same chain starting from two different states a and b. Under some adequate hypothesis, the two chains meet with probability one, and from then on, they start anew and become indistinguable in law; the ergodic principle follows.

Using his coupling method Doeblin proves in Sect. XXII that the set of functions $F(x, y, s, t)$ are uniformly continuous when a, σ and $1/\sigma$ are uniformly bounded. Theorem XXII is one of the keys of the theorems of existence of the regular movements.

In Sect. XXIII, Doeblin aims at showing that $F(x, y, s, t)$ is non increasing in x. He begins with two independent movements with law F, one of which starts from x, and the other from $x' < x$. Between times s and t, the two movements may cross, or remain constantly one below the other; in the first case, their probability to remain below y is the same, whereas in the second case, the lower movement has more chances to remain below y than the upper one, hence the result. Integrating by parts (§ XXIV), it follows that $G(x, y, t, s) = 1 - F(x, y, s, t)$ is the law of a regular movement for the reversed time (see McKean 1969, p. 58, Problème 4, for a similar method and result).

It took a long time (35 years?) for Doeblin's coupling method to be (re)discovered and used, either in the discrete time, or for diffusions (see Pitman 1976; Lindvall 1977, 1983, 1992; Brémaud 1999; Thorisson 2000).

5.5 Other important results of the Pli, which are not reproduced here

To keep this account of the Pli within reasonable length, and to avoid an often high level of technicality, we have not reproduced here the most advanced results of Doeblin about diffusions; we refer the reader to [14], but, nonetheless, here is a brief summary:

Existence theorems

Theorem XXV. If a, σ and $1/\sigma$ are bounded and continuous, there exists a regular solution of Kolmogorov's problem for the coefficients a and σ.

Extensions of Theorem XXV

- If a and σ are continuous, and σ is bounded and vanishes only for a finite number of time values, there exists a regular solution of Kolmogorov's problem for the coefficients a and σ.
- If a and σ are continuous, if for a dense set of values of s, $\sigma(x,s)$ is different from 0 for every x, and the coefficients a and σ are limits of bounded continuous coefficients for which the laws of the associated regular movements are uniformly tight, then there exists a regular solution of Kolmogorov's problem for the coefficients a and σ.

Uniqueness theorem (in the manner of Kolmogorov 1933a, §3)

Theorem XIX. *If a, σ and $1/\sigma$ are continuous, and if there exist a regular solution of Kolmogorov's problem, which is sufficiently differentiable, it is unique and it satisfies the first equation of Kolmogorov with coefficients a and σ.*

Absorption theorem (in the manner of Khinchin (1933, Chap. III, §2)

Theorem XVII. *If a, σ and $1/\sigma$ are continuous, if $v(x,s)$ denotes the probability for a regular movement starting from (x,s) to reach the left part of a given contour C before it reaches its right part, $v(x,s)$ is the solution of the first Kolmogorov's equation, which is 0 on the right and equal to 1 on the left of the contour C (time being represented on the vertical axis).*

Theorems for large values, XIII and XIII' (in the manner of Doeblin) (statements too long to be written completely). For uniformly bounded coefficients, the law of the max of $|X(t) - X(s)|$ is controlled by the law of the max of the absolute amplitude of a standard Brownian motion.

Corollaries XIII and XIII$'$. Very general criteria for non-explosion which extend those of Bernstein (1938) and Doeblin [CR10].

Bibliography

Works of Wolfgang Doeblin

For the reader's ease, we have adopted T. Lindvall's typographical conventions as well as his numbering: [i] for the memoirs and [CRi] for the CRAS Notes (Lindvall 1991).

Memoirs

1. Le cas discontinu des probabilités en chaîne. *Publ. Fac. Sci. Univ. Masaryk* **236**, 3–13, 592 (1937)
2. Sur le cas continu des probabilités en chaîne. *Rend. Accad. Lincei* **25**, 170–176 (1937)

3. Sur les chaînes à liaisons complètes (avec R. FORTET). *Bull. Soc. Math. France* **65**, 132–148 (1937)
4. Sur l'équation de Smoluchowsky. *Praktika de l'Académie d'Athènes* **12**, 116–119 (1937)
5. Sur les propriétés asymptotiques de mouvements régis par certains types de chaînes simples. *Thèse Sci. Math. Paris, Bucarest: Imprimerie Centrale*, 1938, et *Bull. Soc. Math. Roumaine Sci.* **39 (1)**, 57–115, **39 (2)** 3–61 (1937)
6. Sur l'équation matricielle A(t+s)=A(t)A(s) et ses applications aux probabilités en chaîne. *Bull. Sci. Math.* **52**, 21–32, (1940), **64** 35–37 (1938)
7. Sur deux problèmes de M. Kolmogoroff concernant les chaînes dénombrables. *Bull. Soc. Math. France* **66**, 210–220 (1938)
8. Exposé de la Théorie des chaînes simples constantes de Markoff à un nombre fini d'états. *Rev. Math. Union Interbalkanique* **2**, 77–105 (1938)
9. Sur les sommes d'un grand nombre de variables indépendantes. *Bull. Sci. Math.* **63**, 23–32, 35–64 (1939)
10. Sur certains mouvements aléatoires discontinus. *Skand. Aktuarietidskr.* **22**, 211–222 (1939)
11. Remarques sur la théorie métrique des fractions continues. *Compositio Math.* **7**, 353–371 (1940)
12. Éléments d'une théorie générale des chaînes simples constantes de Markoff. *Ann . Sci. École Norm. Sup.* **63**, 317–350 (1940)
13. Sur l'ensemble des puissances d'une loi de probabilité. *Studia Math.* **9**, 71–96 (1940). Reproduit avec un supplément dans *Ann. École Normale Sup.* **63**, 317–350 (1947)
14. Sur l'équation de Kolmogoroff, Pli cacheté déposé le 26 février 1940, ouvert le 18 mai 2000. *C. R. Acad. Sci. Paris*, Série I, **331**, 1031–1187 (2000)

CRAS Notes

[CR1] Sur les sommes de variables aléatoires indépendantes à dispersions bornées inférieurement, (avec P. LÉVY), *C. R. Acad. Sci. Paris*, 202 (1936) pp. 2027–2029.

[CR2] Sur les chaînes discrètes de Markoff, *C. R. Acad. Sci. Paris*, 203 (1936) pp. 24–26.

[CR3] Sur les chaînes de Markoff, *C. R. Acad. Sci. Paris*, 203 (1936) pp. 1210–1211.

[CR4] Sur deux notes de MM. Kryloff et Bogoliouboff, (avec R. FORTET), *C .R. Acad. Sci. Paris*, 204 (1937), pp. 1699–1701.

[CR5] Éléments d'une théorie générale des chaînes constantes simples de Markoff, *C . R. Acad. Sci. Paris*, 205 (1937) pp. 7–9.

[CR6] Premiers éléments d'une étude systématique de l'ensemble des puissances d'une loi de probabilités, *C. R. Acad. Sci. Paris*, 206 (1938) pp. 306–308.

[CR7] Etude de l'ensemble des puissances d'une loi de probabilités, *C. R. Acad. Sci. Paris*, 206 (1938) pp. 718–720.

[CR8] Sur les sommes d'un grand nombre de vecteurs aléatoires, *C. R. Acad. Sci. Paris*, 207 (1938) pp. 511–513.

[CR9] Sur l'équation de Kolmogoroff, *C. R. Acad. Sci. Paris*, 207 (1938) pp. 705–707.

[CR10] Sur certains mouvements aléatoires, *C. R. Acad. Sci. Paris*, 208 (1939) pp. 249–250.

[CR11] Sur un problème de calcul des probabilités, *C. R. Acad. Sci. Paris*, 209 (1939) pp. 742–743.

[CR12] Sur l'équation de Kolmogoroff, *C. R. Acad. Sci. Paris*, 210 (1940) pp. 365–367.

[CR13] Sur des mouvements mixtes, *C. R. Acad. Sci. Paris*, 210 (1940) pp. 690–692.

Other References

Bernstein, S.: Sur l'extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes. Math. Ann. **97**, 1–59 (1926–1927)

Bernstein, S.: Sur les liaisons entre les variables aléatoires. Actes Cong. Int. Math. Zürich **I**, 288–309 (1932)

Bernstein, S.: Équations différentielles stochastiques, Les fonctions aléatoires, Actes du Colloque consacré à la théorie des probabilités et présidé par M. Maurice Fréchet, Genève 11 au 16 octobre 1937, cinquième partie. Paris: Hermann, (Actualités Sci. Ind. 738) 1938, pp. 5–31

Brémaud, P.: Markov chains, Gibbs fields, Monte-Carlo simulation and queues (Texts in Applied Math., No. 31) Berlin Heidelberg New York: Springer 1999

Brockmeyer, E., Halstrøm H. L., Jensen A.: The life and works of A. K. Erlang. Copenhagen: The Copenhagen Telephone Company 1948

Chung, K. L.: The general theory of Markov processes according to Doeblin. Zeit. für Wahr. **2**, 230–254 (1964)

Chung, K. L.: Reminiscences of some of Paul Lévy's ideas in brownian motion and in Markov chains. Seminar on Stochastic Processes 1988. Basel: Birkhäuser 1989, pp. 99–108

Chung, K. L.: Doeblin's Big Limit Theorem. Theo. Prob. **6**, 417–426 (1992)

Chung, K. L.: Reminiscences of one of Doeblin's papers. (Cohn 1993)

Cohn, H. (ed.): Doeblin and modern probability, Blaubeuren 1991 (Contemporary Mathematics 149) Providence: AMS 1993

Cramér, H.: On the mathematical theory of risk. Särrtryck ur Försäkringsaktiebolaget Skandias Festkrift. Stockholm: Centraltryckeriet 1930

Dambis, K. E.: On the decomposition of continuous martingales. Theor. Prob. Appl. **10**, 401–410 (1965)

Döblin, A.: Berlin Alexanderplatz, Olten: Walter-Verlag 1929, 1961

Döblin, A.: Hamlet oder die lange Nacht nimmt ein Ende. Olten: Walter-Verlag 1966

Döblin, A.: Briefe. Olten: Walter-Verlag 1970

Döblin, A.: Autobiographische Schriften und letzte Aufzeichnungen. Olten: Walter-Verlag 1980 (A destiny's journey, Paragon House)

Doob, J. L.: Regularity properties of certain families of chance variables. Trans. Amer. Math. Soc. **47**, 455–486 (1940)

Doob, J. L.: Stochastic processes. New York: Wiley 1953

Doob, J. L.: William Feller and twentieth century probability. Sixth Berkeley Symposium, 1970, vol. 1. Berkeley: University of California Press 1971 pp.XV-XX

Dubins, L., Schwarz G.: On continuous martingales. Proc. Nat. Acad. Sci. USA **53** , 913–916 (1965)

Dynkin, E. B.: Markov processes, vol. 2. Berlin Heidelberg New York: Springer 1965

Einstein, A.: (1905/1926) Investigations on the theory of the brownian movement, ed. with notes by R. Fürth. Translated by A. D. Cooper. New York: Dover 1956

Feller, W.: Zur Theorie der stochastischen Prozesse (Existenz- und Eindeutigkeitssätze). Math. Ann. **113**, 113–160 (1936)

Feller W.: An Introduction to probability theory and its applications, vol. 2. New York: Wiley 1966 (2nd edn. ibid., 1971)

Fisher, R. A.: The genetical theory of natural selection. Oxford: Oxford University Press 1930

Fortet, R.: Sur des fonctions aléatoires définies par leurs équations aux dérivées partielles. C. R. Acad. Sci. Paris **212**, 325–326 (1941)

Fortet, R.: Sur le calcul de certaines probabilités d'absorption. C. R. Acad. Sci. Paris **212**, 1118–1120 (1941)

Fortet, R.: Sur la résolution des équations paraboliques linéaires. C. R. Acad. Sci. Paris **213**, 553–556 (1941)

Fortet, R.: Les fonctions aléatoires du type de Markoff associées à certaines équations aux dérivées partielles de type parabolique. J. Math. Pures Appl. **22** 177–243 (1943)

Fréchet, M.: Théorie des évènements en chaîne dans le cas d'un nombre fini d'états possibles, (tome 1 fascicule 3, Traité du calcul des probabilités et ses applications par Émile Borel). Paris: Gauthier-Villars 1938 (2nd edn. with Supplement and a Note of P. Lévy ibid., 1952)

Fréchet, M.: Les contributions françaises récentes au calcul des probabilités et à la statistique mathématique. Association française pour l'avancement des sciences. Congrès de la Victoire, 20-26 octobre 1945, Mathématiques, reproduit dans Intermédiaire des Recherches Mathématiques, Supplément au fascicule **9**, 107–120 (1947)

Hostinský, B.: Sur les probabilités relatives aux transformations répétées. C. R. Acad. Sci. Paris **186**, 59–61 (1928a)

Hostinský, B.: Compléments à la note sur les probabilités relatives aux transformations répétées. C. R. Acad. Sci. Paris **186**, 487–489 (1928b)

Hostinský, B.: Méthodes générales du calcul des probabilités. Mémorial des sciences mathématiques, fascicule 52. Paris: Gauthier-Villars 1931

Ikeda, N., Watanabe S.: Stochastic differential equations and diffusion processes. Amsterdam: North Holland 1981 (2nd ed. ibid., 1988)

Ikeda, N., Watanabe S., Kunita H., Fukushima M.: Itô's stochastic calculus and probability theory. Berlin Heidelberg New York: Springer, 1996

Itô, K.: Collected works. Berlin Heidelberg New York: Springer, 1987

Itô, K.: Differential equations determining a Markoff process. J. Pan-japan Math. Colloq. **1077**, 1352–1400, (in Japanese) (1942a)

Itô, K.: On stochastic processes (I) (Infinitely divisible laws of probability). Japan J. Math. **18**, 261–301 (1942b)

Itô, K.: Stochastic integral. Proc. Imperial Acad. Tokyo **20**, 519–524 (1944)

Itô, K.: Stochastic differential equations in a differentiable manifold. Nagoya Math. Journ. **1**, 35–47 (1950)

Itô, K.: On stochastic differential equations. Mem. Amer. Math. Soc. **4**, (1951a)

Itô, K.: On a formula concerning stochastic differentials. Nagoya Math. Journ. **3**, 55–65 (1951b)

Itô, K.: Poisson point processes attached to Markov processes. Proc. 6th Berkeley Symposium Math. Stat. Proba., vol. 3. Berkeley: University of California, 1970, pp. 225–239

Itô, K.: Foreword, Collected Works, pp. XIII-XVII, 1987

Itô, K., McKean H. P. Jr: Diffusion processes and their sample paths. New York: Academic Press 1965 (2nd printing corrected, ibid., 1974)

Kac, M.: On distributions of certain Wiener functionals. Trans. Amer. Math. Soc. **65**, 1–13 (1949)

Kahane, J-P.: A century of interplay between Taylor series, Fourier series and Brownian Motion. Bull. London Math. Soc. **29**, 257–279 (1997)

Kahane, J-P.: Le mouvement brownien: un essai sur la théorie mathématique. In: Matériaux pour l'histoire des mathématiques au XXe siècle. Actes du colloque à la mémoire de Jean Dieudonné, (Nice 1996), volume 3 de Séminaires et Congrès, Paris 1998

Kendall, D. G.: Obituary: Andrei Nikolaevich Kolmogorov (1903-1987). Organized by D. G. Kendall. Bull. London Math. Soc. **22**, 31–100 (1990)

Khinchin, A. Y.: Asymptotische Gesetze der Warsheinlichkeitsrechnung. Berlin Heidelberg New York: Springer 1933a (Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, no. 4) (Reprint, New York: Chelsea, 1948)

Kolmogorov, A. N.: Selected Works, 3 vols. Dordrecht: Kluwer Academic Publishing 1991–1993

Kolmogorov, A. N.: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung. Math. Ann. **104**, 149–160 (1931) (English transl. Selected works. II, pp.62–108)
Kolmogorov, A. N.: Zur Theorie der stetigen zufälligen Prozesse. Math. Ann. **108**, 415–458 (1933a) (English transl. Selected works. II, pp. 156–168)
Kolmogorov, A. N.: Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin: Springer (Ergebnisse der Mathematik und ihrer Grenzgebiete vol. 2, no. 3), 1933b. English translation, Foundations of the Theory of Probability, New York: Chelsea, 1950
Kolmogorov, A. N.: Zur Theorie der Markoffschen Ketten, Math. Ann. **112**, 155–160 (1936)
Kunita, H., Watanabe, S.: On square integrable martingales. Nagoya Math. J. **30**, 209–245 (1967)
Lamperti, J.: Probability. A survey of the mathematical theory. W. A. Benjamin, 1966
Lamperti, J.: Stochastic processes. A survey of the mathematical theory. (Applied Mathematics Sciences, vol 23) Berlin Heidelberg New York: Springer 1977
Lévy, P.: Oeuvres complètes, 6 vols. Paris: Gauthier-Villars, 1973–1980
Lévy, P.: Sur les intégrales dont les éléments sont des variables aléatoires indépendantes. Ann. J. Sc. Norm. Sup. Pisa S.**2, (3)**, 337–366, (1934), **4**, 217–218 (1935)
Lévy, P.: Théorie de l'addition des variables aléatoires, (Fascicule I de la Collection des monographies des probabilités, publiée sous la direction de M. Émile Borel). Paris: Gauthier-Villars 1937 (2nd edn. ibid., 1954)
Lévy, P.: Sur certains processus stochastiques homogènes. Compositio Mathematica **7** 283–339 (1939)
Lévy, P.: Le mouvement brownien plan. Amer. J. of Math. **62**, 487–550 (1940)
Lévy, P.: Intégrales stochastiques. C. R. Acad. Sci. Paris **212**, 1066–1068 (1941)
Lévy, P.: Une propriété d'invariance projective dans le mouvement brownien, (pli cacheté déposé à l'Académie des sciences le 16 juin 1943, aux soins de M. Piron, Institut polytechnique de Grenoble, ouvert à la demande de l'auteur le 23 octobre 1944). C. R. Acad. Sci. Paris **219**, 376–378 (1944)
Lévy, P.: Processus stochastiques et Mouvement brownien (Fascicule VI de la Collection des monographies des probabilités, publiée sous la direction de M. Émile Borel), Paris: Gauthier-Villars 1948 (2nd edn. ibid, 1954)
Lévy, P.: Wolfgang Doeblin (V. Doblin) (1915-1940). Rev. Histoire Sci., pp. 107–115 (1955)
Lévy, P.: Le dernier manuscrit inédit de Wolfgang Doeblin. Bull. Sci. Math. **80**, 4 (1956)
Lévy, P.: Quelques aspects de la pensée d'un mathématicien. Paris: Blanchard 1970
Lindvall, T.: Probabilistic proof of Blackwell's renewal theorem. Ann. Prob. **5**, 482–485 (1977)
Lindvall, T.: On coupling of diffusion processes. J. Appl. Proba. **20**, 82–93 (1983)
Lindvall, T.: W. Doeblin 1915–1940. Ann. Prob. **19**, 929–934 (1991)
Lindvall, T.: Lectures on the coupling method. New York: Wiley 1992
Lindvall, T.: Sannolikheten och ödet. Om matematikern Wolfgang Doeblin, Ord & Bild **6**, 48–57 (1993)
Locker, B.: L'intégrale stochastique de Paul Lévy, la vie et l'oeuvre d'un mathématicien juif français sous l'Occupation. Thèse Université Paris **5**, (2001)
Loève, M.: Theory of probability. New-York: Wiley 1955, 1960, 1963, 4th edn. vol. 2. New York: Springer 1977
McKean, H.P.Jr.: Stochastic integrals. New York: Academic Press 1969
Meyer, P.-A.: Les processus stochastiques de 1950 à nos jours. Pier 2000, pp. 813–848
Nelson, E.: Dynamical theories of Brownian motion. Princeton University Press 1967
Neveu, J.: Sur une hypothèse de Feller à propos de l'équation de Kolmogoroff, C. R. Acad. Sci. Paris **240**, 590–591 (1955a)
Neveu, J.: Théorie des semi-groupes de Markoff, Thèse sci. math. Paris 1955b; University of California Publications in Statistics, 1958
Ornstein, L. S., Uhlenbeck, G. E.: On the theory of brownian motion. Phys.Rev. **36**, 823–841 (1930)
Pier, J.-P.: (ed.): Developments of Mathematics 1900-1950. Basel: Birkhäuser 1994
Pier, J.-P.: (ed.): Developments of Mathematics 1950-2000. Basel: Birkhäuser 2000
Pitman, J.: Uniform rates of convergence for Markov chains transition probabilities. Zeit. für Wahr. **29**, 193–227 (1976)
Rogers, L. C. G.: Williams characterization of the Brownian excursion law: proof and applications, Sém Proba. XV, Lect. Notes in Maths, vol. 850. Berlin Heidelberg: Springer 1981, pp. 227–250
Rogers, L. C. G.: A guided tour through excursions. Bull. London Math. Soc. **21**, 305–341 (1989)
Rogers, L. C. G., Wiliams D.: Diffusions, Markov processes, and martingales, Vol. 2: Itô Calculus.

New York: Wiley 1987, (New edition in Cambridge University Press 2000)
Schrödinger, E.: Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique. Ann. Inst. H. Poincaré **2**, 269–310 (1932)
Schrödinger, E.: Statistical thermodynamics. Cambridge: Cambridge University Press 1946-1952 (Rééd. New York: Dover, 1989)
Shiryaev, A. N.: Kolmogorov: life and creative activities. Ann. Prob. **17**, 866–944 (1989)
Shiryaev, A. N.: Essentials of stochastic finance. Facts, models, theory, translated from the russian by N. Kruzhilin. Singapore: World Scientific 1999
Siegmund-Schultze, R.: Rockefeller and the internationalization of mathematics between the two world wars. Basel: Birkhäuser 2001
Stroock, D.: Lectures on stochastic analysis: Diffusion theory, London Math. Soc. Students Texts, 6. Cambridge: Cambridge University Press 1987
Stroock, D.: Probability theory. An analytic view. Cambridge: Cambridge University Press 1993
Stroock, D., Varadhan, S.: Diffusion processes with continuous coefficients I, II. Comm. Pure Appl. Math. **22**, 345–400, 479–530 (1969)
Stroock, D., Varadhan, S.: Multidimensional diffusion processes. Berlin Heidelberg New York: Springer 1979 (2nd edn., ibid., 1997)
Stroock, D., Varadhan, S.: Introduction. (Itô, Collected works), pp. VII-XII
Taqqu, M. S.: Bachelier and his time. Finance Stochast **5**, (1), 3–32 (2001)
Thorrison, H.: Coupling, Stationarity and Regeneration. Berlin Heidelberg New York: Springer 2000
Van Schuppen, J. H., Wong E.: Transformation of local martingales under a change of law. Ann. Prob. **2**, 879–888 (1974)
Varadhan, S. R. S.: Diffusion processes. In: Shanbhag, D. N., Rao, C. R. Handbook of statistics. (eds.) Vol. 19. Elsevier 2001, pp. 853-872
Ventsel, A. D.: Analytical methods in probability theory (Kolmogorov, Selected works), vol. II., pp. 522–527
Ville, J.: Étude critique de la notion de collectif, Thèse sci. math. Paris, (fascicule III de la Collection de monographies des probabilités, publiée sous la direction de M. Émile Borel). Paris: Gauthier-Villars, 1939
Volkonskii, V.A.: Random substitution of time in strong Markov processes. Teor. Veroyatnost. i. Primenen **3**, 332–350 (1958)
Watanabe, S.: Itô's stochastic calculus and its applications. In: Shanbhag, D.N., Rao, C.R. (eds.) Handbook of Statistics. vol. 19. Elsevier 2001, pp. 873–934
Williams, D.: Diffusions, Markov processes and martingales 1: Foundations. New York Wiley 1979